

Minimum degree condition for spanning generalized Halin graphs

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Abstract. A spanning tree with no vertices of degree 2 is called a Homeomorphically irreducible spanning tree (HIST). Based on a HIST embedded in the plane, a Halin graph is formed by connecting the leaves of the tree into a cycle following the cyclic order determined by the embedding. Both of the determination problems of whether a graph contains a HIST or whether a graph contains a spanning Halin graph are shown to be NP-complete. It was conjectured by Albertson, Berman, Hutchinson, and Thomassen in 1990 that a *every surface triangulation of at least four vertices contains a HIST* (confirmed). And it was conjectured by Lovász and Plummer that *every 4-connected plane triangulation contains a spanning Halin graph* (disproved). Balancing the above two facts, in this paper, we consider generalized Halin graphs, a family of graph structures which are “stronger” than HISTs but “weaker” than Halin graphs in the sense of their construction constraints. To be exact, a generalized Halin graph is formed from a HIST by connecting its leaves into a cycle. Since a generalized Halin graph needs not to be planar, we investigate the minimum degree condition for a graph to contain it as a spanning subgraph. We show that there exists a positive integer n_0 such that any 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n + 3)/5$ contains a spanning generalized Halin graph. As an application, the result implies that under the same condition, the graph G contains a wheel-minor of order at least $n/2$. The minimum degree condition in the result is best possible.

Keywords. Homeomorphically irreducible spanning tree; Halin graph; Hamiltonian cycle

1 Introduction

A tree with no vertex of degree 2 is called a *homeomorphically irreducible tree (HIT)*, and a spanning tree with no vertex of degree 2 is a *homeomorphically irreducible spanning tree (HIST)*. A *Halin graph*, constructed by Halin in 1971 [9], is a graph formed from a plane embedding of a HIST by connecting its leaves into a cycle following the cyclic order determined by the embedding. In 1990, Albertson, Berman, Hutchinson, and Thomassen [1] showed that *it is NP-complete to determine whether a graph contains a HIST*. However, for special graph classes such as triangulations of surfaces, they conjectured that *every triangulation of a surface with at least 4 vertices contains a HIST*. The conjecture was confirmed in [6]. It was shown by Horton, Parker, and Borie [10] that *it*

is NP-complete to determine whether a graph contains a (spanning) Halin graph. Again, restricted to triangulations, Lovász and Plummer [14] conjectured that *every 4-connected plane triangulation contains a spanning Halin graph*. But the conjecture was disproved recently [5]. Since a Halin graph possesses many hamiltonian properties (e.g., see [3, 7, 4]), it seems that a graph has to have very “good properties” in order to contain a Halin graph as a spanning subgraph. For this reason, by relaxing on the planarity requirement, we define a *generalized Halin graph* as a graph formed from a HIST by connecting its leaves into a cycle, and we study sufficient conditions for implying the containment of a spanning generalized Halin graph in a given graph.

Compared to Halin graphs, generalized Halin graphs are less studied. Kaiser et al. in [11] showed that a generalized Halin graph is *prism Hamiltonian*; that is, the Cartesian product of a generalized Halin graph and K_2 is hamiltonian. Since a tree with no degree 2 vertices has more leaves than the non-leaves, a generalized Halin graph contains a cycle of length at least half of its order. Also, one can notice that by contracting the non-leaves of the underlying tree of a generalized Halin graph into a single vertex, a wheel graph is resulted with the contracted vertex as the hub, where a minor of a graph is obtained from the graph by deleting edges/contracting edges, or deleting vertices. Therefore, a generalized Halin graph contains a wheel-minor of order at least half of its order. The investigation on the properties of generalized Halin graphs is not of the interest of this paper. Instead, in this paper, we show the following two results.

Theorem 1.1. *It is NP-complete to determine whether a graph contains a spanning generalized Halin graph.*

Theorem 1.2. *There exists a positive integer n_0 such that every 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n + 3)/5$ contains a spanning generalized Halin graph. The result is best possible in the sense of the connectivity and minimum degree constraints.*

Since a generalized Halin graph of order n contains a wheel-minor of order at least $n/2$, we get the following corollary.

Corollary 1.1. *There exists a positive integer n_0 such that every 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n + 3)/5$ contains a wheel-minor of order at least $n/2$.*

For notational convenience, for a graph T , we denote by $L(T)$ the set of degree 1 vertices of T and $S(T) = V(T) - L(T)$. Also we abbreviate *spanning generalized Halin graph* as *SGHG* in what follows, and denote a generalized Halin graph as $H = T \cup C$, where T is the underlying HIST of H and C is the cycle spanning on $L(T)$. The remaining of the paper is organized as follows. In Section 2, we prove Theorem 1.1 and show the sharpness of Theorem 1.2. In Section 3, we introduce some notations and lemmas, which are used in the proof of Theorem 1.2. We then proof Theorem 1.2 in Section 4.

2 Proof of Theorem 1.1 and the sharpness of Theorem 1.2

Proof of Theorem 1.1. It was shown by Albertson et al. [1] that it is NP-complete to decide whether a graph contains a HIST, and by the definition, a generalized Halin graph contains a HIST. Hence, we see that the problem of deciding whether an arbitrary graph contains an SGHG is in NP. To show the problem is NP-complete we assume the existence of a polynomial algorithm to test for an SGHG and use it to create a polynomial algorithm to test for a hamiltonian path between two vertices in an arbitrary graph. The decision problem for such hamiltonian paths is a classic NP-complete problem [8].

Let G be a graph and $x, y \in V(G)$. We want to determine whether there exists a hamiltonian path connecting x and y . We first construct a new graph G' and show that G contains a hamiltonian path between x and y if and only if G' contains a HIST (the proof of this part is the same as the proof of Albertson et al. in [1]). Then based on G' , we construct a graph G'' and show that G' contains a HIST if and only if G'' contains an SGHG.

Let $\{z_1, z_2, \dots, z_t\} = V(G) - \{x, y\}$. Then G' is formed by adding new vertices $\{z'_1, z'_2, \dots, z'_t\}$ and new edges $\{z_i z'_i : 1 \leq i \leq t\}$. It is clear that if P is a hamiltonian path between x and y , then $P \cup \{z_i z'_i : 1 \leq i \leq t\}$ is a HIST of G' . Conversely, let T be a HIST of G' . Since $1 \leq d_T(z'_i) \leq d_{G'}(z'_i) = 1$, we get $d_T(z'_i) = 1$ for each i . Since $N_{G'}(z'_i) = \{z_i\}$ and T is a HIST, we have $d_T(z_i) \geq 3$. Hence $T - \{z'_1, z'_2, \dots, z'_t\}$ is a tree with leaves possibly in $\{x, y\}$. Since each tree has at least 2 leaves and a tree with exactly two leaves is a path, we conclude that $T - \{z'_1, z'_2, \dots, z'_t\}$ is a path between x and y .

Then based on G' , we construct a graph G'' . First we add new vertices $\{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$. Then we add edges $\{z'_i z'_{i1}, z'_i z'_{i2}, z'_i z'_{i3}, z'_{i1} z'_{i2}, z'_{i2} z'_{i3} : 1 \leq i \leq t\}$. Finally, we connect all vertices in $\{x, y\} \cup \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$ into a cycle C'' such that $\{z'_{i1} z'_{i2}, z'_{i2} z'_{i3} : 1 \leq i \leq t\} \subseteq E(C'')$. If T' is a HIST of G' , then $T'' := T' \cup \{z'_i z'_{i1}, z'_i z'_{i2}, z'_i z'_{i3} : 1 \leq i \leq t\}$ is a HIST of G'' and $T'' \cup C''$ is an SGHG of G'' . Conversely, suppose $H = T \cup C$ is an SGHG of G'' . We claim that $C = C''$. This in turn gives that $T = T''$ and therefore $T'' - \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$ is a HIST of G' . To show that $C = C''$, we first show that $z'_{i2} \in L(T)$ for each i . Suppose on the contrary and assume, without loss of generality, that $z'_{12} \in S(T)$. Then as $N_{G''}(z'_{12}) = \{z'_1, z'_{11}, z'_{13}\}$, we get $\{z'_{12} z'_1, z'_{12} z'_{11}, z'_{12} z'_{13}\} \subseteq E(T)$. Since T is acyclic, $z'_{11} z'_1, z'_{13} z'_1 \notin E(T)$. This in turn shows that $\{z'_1, z'_{11}, z'_{13}\} \subseteq L(T)$. However, $\{z'_{12} z'_1, z'_{12} z'_{11}, z'_{12} z'_{13}\}$ forms a component of T , showing a contradiction. Then we show that $z'_{i1}, z'_{i3} \in L(T)$ for each i . Suppose on the contrary and assume, without loss of generality, that $z'_{11} \in S(T)$. By the previous argument, we have $z'_{12} \in L(T)$. Then $z'_1, z'_{13} \in L(T)$ as z'_{12} is on C and z'_1 and z'_{13} are the only two neighbors of z'_{12} which can be on the cycle C . As $d_{G''}(z'_{11}) = 3$ and $\{z'_{12}, z'_1\} \subseteq N_{G''}(z'_{11})$, $z'_{11} z'_{12}, z'_{11} z'_1 \in E(T)$. Since $z'_{12} \in L(T)$ and $z'_1, z'_{13} \in L(T)$, we get $z'_{12} z'_{13}, z'_{12} z'_1, z'_1 z'_{13} \notin E(T)$. Since $d_{G''}(z'_{12}) = d_{G''}(z'_{13}) = 3$, we have $z'_{12} z'_{13}, z'_{12} z'_1, z'_1 z'_{13} \in E(C)$. However, $z'_{12} z'_{13}, z'_{12} z'_1, z'_1 z'_{13}$ forms a triangle but $|V(C)| \geq 4$, showing a contradiction. So we have shown that $\{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\} \subseteq L(T)$. This indicates that in the tree $T - \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$, each vertex z'_i has degree 1 and no vertices of degree 2. Hence

$T - \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$ is a HIST of G' .

Combining the arguments in the two paragraphs above, we see that G has a hamiltonian path between x and y if and only if G'' has an SGHG. Hence a polynomial SGHG-tester becomes a polynomial path-tester. \blacksquare

Since a generalized Halin graph is 3-connected, the connectivity requirement in Theorem 1.2 is necessary. To show that the minimum degree requirement is best possible, we show the following proposition.

Proposition 1. *Let $G(A, B) = K_{a,b}$ be a complete bipartite graph with $|A| = a$ and $|B| = b$. Then $G(A, B)$ has no HIST T with $|L(T) \cap A| = |L(T) \cap B|$ if $b > \frac{3(a-1)}{2}$.*

If a bipartite graph $G(A, B)$ contains an SGHG $H = T \cup C$, then $|L(T) \cap A| = |L(T) \cap B|$. Thus, by Proposition 1, it is easy to see that the complete bipartite graphs $K_{a,b}$ with $b = \frac{3a-1}{2}$ when a is odd and $b = \frac{3a-2}{2}$ when a is even does not have an SGHG. Let $n = a + b$. By direct computation, we get $\delta(K_{a,b}) = \frac{2n+1}{5}$ when $b = \frac{3a-1}{2}$ and $\delta(K_{a,b}) = \frac{2n+2}{5}$ when $b = \frac{3a-2}{2}$. We now prove Proposition 1.

Proof of Proposition 1. Suppose on the contrary that $G(A, B)$ contains a HIST T such that $|L(T) \cap A| = |L(T) \cap B|$. Then

$$\begin{aligned} |S(T) \cap B| - |S(T) \cap A| &= |B| - |L(T) \cap B| - (|A| - |L(T) \cap A|) \\ &= |B| - |A| > \frac{3(a-1)}{2} - a = \frac{a-3}{2}. \end{aligned}$$

Since $G(A, B)$ is bipartite and T is a HIST of $G(A, B)$, we have $|S(T) \cap A| \geq 1$. Thus, from the inequalities above, we obtain $|S(T) \cap B| > (a-1)/2$. Since T is a HIST, we have $d_T(y) \geq 3$ for each $y \in S(T) \cap B$. Let $E_B = \{e \in E(T) : e \text{ is incident to a vertex in } S(T) \cap B\}$. Denote by T' the subgraph of T induced on E_B . Notice that T' is a forest of at least $3|S(T) \cap B|$ edges. Hence T' has at least $3|S(T) \cap B| + 1$ vertices. As T' is a bipartite graph with one partite set as $S(T) \cap B$, and another as a subset of A , we conclude that $|V(T) \cap A| = |V(T)| - |S(T) \cap B| \geq 2|S(T) \cap B| + 1$. Since $|S(T) \cap B| > (a-1)/2$, we then have $|V(T) \cap A| > a$. This gives a contradiction to the assumption $|A| = a$. \blacksquare

3 Notations and Lemmas

We consider in this paper simple and finite graphs only. Given a graph G , we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G , respectively, and by $e(G)$ the size of G . Let $S \subseteq V(G)$ and $v \in V(G)$. Denote by $G[S]$ the subgraph of G induced on S , and denote by $\Gamma_G(v, S)$ the set of neighbors of v in S , and $\deg_G(v, S) = |\Gamma_G(v, S)|$. When $S = V(G)$, we only write $\Gamma_G(v)$ and $\deg_G(v)$. For two subsets $U_1, U_2 \subseteq V(G)$, let $\delta_G(U_1, U_2) = \min\{\deg_G(u_1, U_2) : u_1 \in U_1\}$ and

$\Delta_G(U_1, U_2) = \max\{\deg_G(u_1, U_2) : u_1 \in U_1\}$. Denote by $E_G(U_1, U_2)$ the set of edges with one end in U_1 and the other in U_2 , the cardinality of $E_G(U_1, U_2)$ is denoted by $e_G(U_1, U_2)$. Let $u, v \in V(G)$ be two vertices. We write $u \sim v$ if u and v are adjacent. A path connecting u and v is called a (u, v) -path. If G is a bipartite graph with partite sets A and B , we denote G by $G(A, B)$ for specifying the two partite sets. A *matching* in G is a set of independent edges; a \wedge -*matching* is a set of vertex-disjoint copies of $K_{1,2}$; and a *claw-matching* is a set of vertex-disjoint copies of $K_{1,3}$. The set of degree 2 vertices in a \wedge -matching is called the center of the \wedge -matching; and the set of degree 3 vertices in a claw-matching is called the center of the claw-matching. A cycle C in a graph G is *dominating* if $G - V(C)$ is an edgeless graph.

The Regularity Lemma of Szemerédi [18] and Blow-up lemma of Komlós et al. [12] are main tools in our proof of Theorem 1.2. For any two disjoint non-empty vertex-sets A and B of a graph G , the *density* of A and B is the ratio $d(A, B) := \frac{e(A, B)}{|A||B|}$. Let ε and δ be two positive real numbers. The pair (A, B) is called ε -regular if for every $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, $|d(X, Y) - d(A, B)| < \varepsilon$ holds. In addition, if $\deg(a, B) > \delta|B|$ for each $a \in A$ and $\deg(b, A) > \delta|A|$ for each $b \in B$, we say (A, B) an (ε, δ) -super regular pair.

Lemma 3.1 (Regularity lemma-Degree form [18]). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if G is any graph with n vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set $V(G)$ into $l + 1$ clusters V_0, V_1, \dots, V_l , and there is a spanning subgraph $G' \subseteq G$ with the following properties.*

- $l \leq M$;
- $|V_0| \leq \varepsilon n$, all clusters $|V_i| = |V_j| \leq \lceil \varepsilon n \rceil$ for all $1 \leq i \neq j \leq l$;
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n$ for all $v \in V(G)$;
- $e(G'[V_i]) = 0$ for all $i \geq 1$;
- all pairs (V_i, V_j) ($1 \leq i < j \leq l$) are ε -regular, each with a density either 0 or greater than d .

Lemma 3.2 (Blow-up lemma-weak version [12]). *Given a graph R of order r and positive parameters δ, Δ , there exists a positive $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds. Let n_1, n_2, \dots, n_r be arbitrary positive integers and let us replace the vertices v_1, v_2, \dots, v_r with pairwise disjoint sets V_1, V_2, \dots, V_r of sizes n_1, n_2, \dots, n_r (blowing up). We construct two graphs on the same vertex set $V = \bigcup V_i$. The first graph K is obtained by replacing each edge $v_i v_j$ of R with the complete bipartite graph between the corresponding vertex sets V_i and V_j . A sparser graph G is constructed by replacing each edge $v_i v_j$ arbitrarily with an (ε, δ) -super regular pair between V_i and V_j . If a graph H with $\Delta(H) \leq \Delta$ is embeddable into K then it is already embeddable into G .*

Lemma 3.3 (Blow-up lemma-strengthened version [12]). *Given $c > 0$, there are positive numbers $\varepsilon = \varepsilon(\delta, \Delta, r, c)$ and $\gamma = \gamma(\delta, \Delta, r, c)$ such that the Blow-up lemma in the equal size case (all $|V_i|$ are the same) remains true if for every i there are certain vertices x to be embedded into V_i whose images are a priori restricted to certain sets $C_x \subseteq V_i$ provided that*

- (i) each C_x within a V_i is of size at least $c|V_i|$;

(ii) the number of such restrictions within a V_i is not more than $\gamma|V_i|$.

We will use both the weak and strengthened versions of Blow-up lemma in our proof.

Besides the above two lemmas, we also need the two lemmas below regarding regular pairs.

Lemma 3.4. *If (A, B) is an ε -regular pair with density d , then for any $A' \subseteq A$ with $|A'| > \varepsilon|A|$, there are at most $\varepsilon|B|$ vertices $b \in B$ such that $\deg(b, A') \leq (d - \varepsilon)|A'|$.*

Lemma 3.5 (Slicing lemma). *Let (A, B) be an ε -regular pair with density d , and for some $\nu > \varepsilon$, let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \nu|A|$, $|B'| \geq \nu|B|$. Then (A', B') is an ε' -regular pair of density d' , where $\varepsilon' = \max\{\varepsilon/\nu, 2\varepsilon\}$ and $d' > d - \varepsilon$.*

The following two results on hamiltonicity are used in finding cycles in the proofs.

Lemma 3.6 ([17]). *If G is a graph of order n satisfying $d(x) + d(y) \geq n + 1$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is hamiltonian-connected.*

Lemma 3.7 ([15]). *Let G be a balanced bipartite graph with $2n$ vertices. If $d(x) + d(v) \geq n + 1$ for any two non-adjacent vertices $x, y \in V(G)$, then G is hamiltonian.*

4 Proof of Theorem 1.2

Given $0 \leq \beta \ll \alpha \ll 1$, we define the two extremal cases with parameters α and β as follows.

Extremal Case 1. There exists a partition of $V(G)$ into V_1 and V_2 such that $|V_i| \geq (2/5 - 4\beta)n$ and $d(V_1, V_2) < \alpha$. Furthermore, $\deg(v_1, V_2) \leq 2\beta n$ for each $v_1 \in V_1$.

Extremal Case 2. There exists a partition of $V(G)$ into V_1 and V_2 such that $|V_1| > (3/5 - \alpha)n$ and $d(V_1, V_2) \geq 1 - 3\alpha$. Furthermore, $\deg(v_1, V_2) \geq (2n + 3)/5 - 2\beta n$ for each $v_1 \in V_1$.

Then Theorem 1.2 is shown through the following three theorems.

Theorem 4.1 (Non-extremal Case). *For every $\alpha > 0$, there exists $\beta > 0$ and a positive integer n_0 such that if G is a 3-connected graph with $n \geq n_0$ vertices and $\delta(G) \geq (2n + 3)/5 - \beta n$, then G contains an SGHG or G is in one of the two extremal cases.*

Theorem 4.2 (Extremal Case 1). *Suppose that $0 < \beta \ll \alpha \ll 1$ and n is a sufficiently large integer. Let G be a 3-connected graph on n vertices with $\delta(G) \geq (2n + 3)/5$. If G is in Extremal Case 1, then G contains an SGHG.*

Theorem 4.3 (Extremal Case 2). *Suppose that $0 < \beta \ll \alpha \ll 1$ and n is a sufficiently large integer. Let G be a 3-connected graph on n vertices with $\delta(G) \geq (2n + 3)/5$. If G is in Extremal Case 2, then G contains an SGHG.*

We show Theorems 4.1-4.3 separately in the following three subsections.

4.1 Proof of Theorem 4.1

We fix the following sequence of parameters,

$$0 < \varepsilon \ll d \ll \beta \ll \alpha < 1, \quad (1)$$

and specify their dependence as the proof proceeds. We let $\beta \ll \alpha$ be the same α and β as defined in the two extremal cases. Then we choose $d \ll \beta$. Finally we choose

$$\varepsilon = \min \left\{ \frac{1}{4}\varepsilon \left(\frac{d}{2}, \left\lceil \frac{2}{d^3} \right\rceil, 2, \frac{d}{2} \right), \frac{1}{9}\varepsilon \left(\frac{d}{2}, \left\lceil \frac{3}{d^3} \right\rceil, 3 \right), \frac{1}{4}\varepsilon \left(\frac{d}{2}, 2, 2, \frac{d}{2} \right) \right\}, \quad (2)$$

where $\varepsilon \left(\frac{d}{2}, \left\lceil \frac{3}{d^3} \right\rceil, 3 \right)$ follows from the definition of the ε in the weak version of the Blow-up lemma and $\varepsilon \left(\frac{d}{2}, \left\lceil \frac{2}{d^3} \right\rceil, 2, \frac{d}{2} \right)$ and $\varepsilon \left(\frac{d}{2}, 2, 2, \frac{d}{2} \right)$ follow from the definition of the ε in the strengthened version of the Blow-up lemma. Choose n to be sufficiently large. In the proof, we omit non-necessary ceiling and floor functions.

Let G be a graph of order n such that $\delta(G) \geq (2n+3)/5 - \beta n$ and suppose that G is not in any of the two extremal cases. Applying the regularity lemma to G with parameters ε and d , we obtain a partition of $V(G)$ into $l+1$ clusters V_0, V_1, \dots, V_l for some $l \leq M = M(\varepsilon)$, and a spanning subgraph G' of G with all described properties in Lemma 3.1 (the Regularity lemma). In particular, for all $v \in V$,

$$\begin{aligned} \deg_{G'}(v) &> \deg_G(v) - (d + \varepsilon)n \geq (2/5 - \beta - d - \varepsilon)n \\ &\geq (2/5 - 2\beta)n \text{ (provided that } \varepsilon + d \leq \beta), \end{aligned} \quad (3)$$

and

$$e(G') \geq e(G) - \frac{(d + \varepsilon)}{2}n^2 \geq e(G) - dn^2,$$

by using $\varepsilon < d$.

We further assume that $l = 2k$ is even; otherwise, we eliminate the last cluster V_l by removing all the vertices in this cluster to V_0 . As a result, $|V_0| \leq 2\varepsilon n$ and

$$(1 - 2\varepsilon)n \leq lN = 2kN \leq n, \quad (4)$$

here we assume that $|V_i| = N$ for $i \geq 1$.

For each pair i and j with $1 \leq i < j \leq l$, we write $V_i \sim V_j$ if $d(V_i, V_j) \geq d$. We now consider the reduced graph G_r , whose vertex set is $\{1, 2, \dots, l\}$, and two vertices i and j are adjacent if and only if $V_i \sim V_j$. We claim that $\delta(G_r) \geq (2/5 - 2\beta)l$. Suppose not, and let $i_0 \in V(G_r)$ such that $\deg(i_0, V(G_r)) < (2/5 - 2\beta)l$. Then, for the corresponding cluster V_{i_0} we have $e_{G'}(V_{i_0}, V(G') - V_{i_0}) < |V_{i_0}|(2/5 - 2\beta)lN$. On the other hand, by using (3), we have $e_{G'}(V_{i_0}, V(G') - V_{i_0}) \geq |V_{i_0}|(2/5 - 2\beta)n$. As $lN \leq n$ from (4), we obtain a contradiction. The rest of the proof consists of the following steps.

Step 1. Show that G_r contains a dominating cycle C and there is a \wedge -matching in G_r with all vertices in $V(G_r) - V(C)$ as its center. We distinguish two cases in Step 1, and each of the other steps will be separated into two cases correspondingly.

Case A. $C = X_1Y_1X_2Y_2 \cdots X_tY_t$ is an even cycle for some $t \leq k$.

Case B. $C = X_0X_1Y_1X_2Y_2 \cdots X_tY_t$ is an odd cycle for some $t < k$.

Notice that in Case B there is at least one vertex in $V(G_r) - V(C)$ by the assumption that $|V(G_r)| = l$ is even. In what follows, if we denote a vertex of G_r by a capital letter, it means either a vertex of G_r or the corresponding cluster in G , but the exact meaning will be clear from the context. For $1 \leq i \leq t$, we call X_i and Y_i the partners of each other, and write as $P(X_i) = Y_i$ and $P(Y_i) = X_i$.

Since C is not necessarily hamiltonian in G_r , we need to take care of the clusters of G which are not represented on C . For each vertex $F \in V(G_r) - V(C)$, we partition the corresponding cluster F into two small clusters F_1 and F_2 such that $-1 \leq |F_1| - |F_2| \leq 1$. We call each F_1 and F_2 a *half-cluster*. Then we group all the original clusters and the partitioned clusters into pairs (A, B) and triples (C, D, F) with F as a half-cluster such that each pair (A, B) and (C, D) is still ε -regular with density d and the pair (D, F) is 2.1ε -regular with density $d - \varepsilon$. Having the cluster groups like this, in the end, we will find “small” HITs within each pair (A, B) or among each triple (C, D, F) .

Step 2. For each $1 \leq i \leq t - 1$, initiate two independent edges connecting Y_i and X_{i+1} . In Case A, also initiate two independent edges connecting X_1 and Y_t ; and in Case B, initiate two independent edges connecting the clusters in each pair of X_0 and X_1 , and X_0 and Y_t .

Step 3. Make each regular pair in the new grouped pairs and triples given in Step 1 super-regular.

Step 4. Construct HITs covering all vertices in V_0 using vertices from the super-regular pairs obtained from Step 3, and obtain new super-regular pairs.

Step 5. Apply the Blow-up lemma to find a HIT between a super-regular pair resulted from Step 4 or among a triple (A, B, F) , where both (A, F) and (A, B) are super-regular pairs resulted from Step 4, and F is a half cluster. In addition, in the construction, for each triple (A, B, F) , we require the HIT to use as many vertices as possible from F as non-leaves.

Step 6. Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two disjoint paths covering all the leaves. Then connect all the HITs into a HIST of G using edges guaranteed by the regularity and connect the disjoint paths into a cycle using the edges initiated in Step 2. The union of the HIST and the cycle gives an SGHG of G .

We now give details of each step. The assumption that G is not in any of the two extremal cases leads to the following claim, which will be used in Step 1.

Claim 4.1. *Each of the following holds for G_r .*

- (a) G_r contains no cut-vertex set of size at most βl ;
- (b) G_r contains no independent set of size more than $(3/5 - \alpha/2)l$.

Proof. (a) Suppose instead that G_r contains a vertex-cut W of size at most βl . As $\delta(G_r) \geq (2/5 - 2\beta)l$, then each component of $G_r - W$ has at least $(2/5 - 3\beta)l$ vertices. Let U be the vertex set of one of the components of $G_r - W$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$. We see that $|A|, |B| \geq (2/5 - 3\beta)lN \geq (2/5 - 4\beta)n$, and since $e(G) \leq e(G') + dn^2$, we have

$$\begin{aligned}
e_G(A, B) &\leq e_{G'}(A, B) + dn^2 \leq |W||A| + dn^2 \\
&\leq \beta lN(3/5 + 3\beta)lN + dn^2 \leq (3\beta/5 + 3\beta^2 + d)n^2 \quad (\text{as } |A| \leq (3/5 + 3\beta)lN \text{ and } ln \leq n) \\
&\leq \frac{25}{3}(3\beta/5 + 3\beta^2 + d)|A||B| \quad (\text{since } |A||B| \geq 3n^2/25) \\
&< \alpha|A||B| \quad (\text{provided that } \frac{25}{3}(3\beta/5 + 3\beta^2 + d) < \alpha).
\end{aligned}$$

This shows that $d(A, B) < \alpha$. Since $\deg_{G_r}(u, V(G_r) - U) = \deg_{G_r}(u, W) \leq \beta l$ for each $u \in U$, we see that $\deg_G(a, B) \leq \beta lN + (d + \varepsilon)n \leq 2\beta n$ for each $a \in A$ provided that $d + \varepsilon \leq \beta$. However, the above argument shows that G is in Extremal Case 1, showing a contradiction.

(b) Suppose instead that G_r contains an independent set U of size larger than $(3/5 - \alpha/2)l$. Let $U' = V(G_r) - U$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$. Then $|A| \geq (3/5 - \alpha/2)lN \geq (3/5 - \alpha)n$. For each vertex $v \in A$, since $\deg_G(v, A) \leq \deg_{G'}(v, A) + (d + \varepsilon)n \leq \beta n$, we have $\deg_G(v, B) \geq (2n + 3)/5 - \beta n - \beta n \geq (2n + 3)/5 - 2\beta n$. This gives that

$$d(A, B) \geq \frac{(2/5 - 2\beta)n}{|B|} \geq \frac{(2/5 - 2\beta)n}{(2/5 + \alpha)n} \geq 1 - 3\alpha,$$

provided that $\beta \leq \alpha/10 + 3\alpha^2/2$. We see that G is in Extremal Case 2. \square

Step 1. Show that G_r contains a dominating cycle C , and there is a \wedge -matching in G_r with all vertices in $V(G_r) - V(C)$ as its center.

We need some results on longest cycles and paths as follows.

Lemma 4.1 ([16]). *Let G be a 2-connected graph on n vertices with $\delta(G) \geq (n + 2)/3$. Then every longest cycle in G is a dominating cycle.*

Lemma 4.2 ([2]). *Let G be a 2-connected graph on n vertices with $\delta(G) \geq (n + 2)/3$. Then G contains a cycle of length at least $\min\{n, n + \delta(G) - \alpha(G)\}$, where $\alpha(G)$ is the size of a largest independent set in G .*

Lemma 4.3 ([13]). *If G is a 3-connected graph of order n such that the degree sum of any four independent vertices is at least $3n/2 + 1$, then the number of vertices on a longest path and that on a longest cycle differs at most by 1.*

By (a) of Claim 4.1, G_r is βl -connected. Since $n = Nl + |V_0| \leq (l + 2)\varepsilon n$, we get $l \geq 1/\varepsilon - 2$. Since $1/\varepsilon - 2 \geq 3/\beta$ (provided that $\beta \geq 3\varepsilon/(1 - 2\varepsilon)$), we then have $\beta l \geq 3$. So G_r is 3-connected.

By Claim 4.1 (b), G_r has no independent set of size more than $(3/5 - \alpha/2)l$. Notice that $\delta(G_r) \geq (2/5 - 2\beta)l > (l + 2)/3$. Applying Lemma 4.1 and Lemma 4.2 on G_r , we see that there is a cycle C in G_r which is longest, dominating, and has length at least $(4/5 + \alpha/2 - 2\beta)l$. Let $\mathcal{W} = V(G_r) - V(C)$. In Case B, we order and label the vertices of C such that X_0 is adjacent to a vertex, say $Y_0 \in \mathcal{W}$ (recall that $\mathcal{W} \neq \emptyset$ in this case). We fix (X_0, Y_0) as a pair at the first place ($X_0 Y_0 \in E(G_r)$), as cluster in G , (X_0, Y_0) is an ε -regular pair with density d . Let

$$\mathcal{W}' = \begin{cases} \mathcal{W}, & \text{if in Case A;} \\ \mathcal{W} - \{Y_0\}, & \text{if in Case B.} \end{cases}$$

We have $|\mathcal{W}'| \leq (1/5 - \alpha/2 + 2\beta)l$ if in Case A and $|\mathcal{W}'| \leq (1/5 - \alpha/2 + 2\beta)l - 1$ if in Case B. So $2|\mathcal{W}'| \leq (2/5 - \alpha + 4\beta)l < (2/5 - 2\beta)l$ (provided that $\beta < \alpha/6$) if in Case A and $2|\mathcal{W}'| \leq (2/5 - \alpha + 4\beta)l - 2 < (2/5 - 2\beta)l - 1$ (provided that $\beta < \alpha/6$) if in Case B. Thus there is a \wedge -matching centered in all vertices in \mathcal{W}' ; furthermore, if in Case B, we can choose the matching such that X_0 is not covered by it. Let M_\wedge be such a matching. For a vertex $X \in \mathcal{W}'$, denote by $M_\wedge(X)$ the two vertices from $V(C)$ to which X is adjacent in M_\wedge . Then we have two facts about vertices in $M_\wedge(X)$.

Fact 1. *Let $X \in \mathcal{W}'$. Then the two vertices in $M_\wedge(X)$ are non-consecutive on C . (By the assumption that C is longest.)*

Fact 2. *Let $X \neq Y \in \mathcal{W}'$. Then no two vertices from $M_\wedge(X) \cup M_\wedge(Y)$ are adjacent on C . (By applying Lemma 4.3.)*

For a complete bipartite graph, if it contains an SGHG, then the ratio of the cardinalities of the two partite sets should be greater than $2/3$ as shown in Proposition 1. Since a longest dominating cycle in G_r is not necessarily hamiltonian, we need to take care of the clusters of G which are not represented by the vertices on C . One possible consideration is that for each $F \in V(G_r) - V(C)$, suppose F is adjacent to $A \in V(C)$, recall $P(A)$ is the partner of A . Then as clusters, we consider the bipartite graph of G with partite sets A and $P(A) \cup F$. However, $|A|/|P(A) \cup F|$ is about $1/2$, which is less than $2/3$. For this reason, we partition $F \in V(G_r) - V(C)$ into two parts to attain the right ratio in the corresponding bipartite graphs. Suppose $M_\wedge(F) = \{D_1, D_2\} \subseteq V(C)$. As a cluster of G , we partition F into F_1 and F_2 arbitrarily such that

$$|F_1| = \left\lfloor \frac{|F|}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor \quad \text{and} \quad |F_2| = \left\lceil \frac{|F|}{2} \right\rceil = \left\lceil \frac{N}{2} \right\rceil.$$

We call each F_i a half-cluster of G . Then we create two pairs (D_i, F_i) , and call D_i the dominator of F_i , and F_i the follower of D_i , and (D_i, F_i) a DF-pair, for $i = 1, 2$. We have the following fact about a DF-pair.

Fact 3. *Each DF-pair (D, F) is 2.1ε -regular with density at least $d - \varepsilon$. (By Slicing lemma.)*

Also, by Fact 1 and Fact 2, if $D \in V(C)$ is a dominator, then $P(D)$, the partner of D , is not a dominator for any followers. As $X_0 \notin V(\mathcal{W}')$, we know that X_0 is not a dominator for any half-clusters. We group the clusters and half-clusters of G into H -pairs and H -triples in a way below.

For each pair (X_i, Y_i) on C , if $\{X_i, Y_i\} \cap V(M_\wedge) = \emptyset$, we take (X_i, Y_i) as an H-pair. Otherwise, $|\{X_i, Y_i\} \cap V(M_\wedge)| = 1$ by Fact 1 and Fact 2. Since there is no difference for the proof for the case that $X_i \in V(M_\wedge)$ or the case that $Y_i \in V(M_\wedge)$, throughout the remaining proof, we always assume that $Y_i \in V(M_\wedge)$ if $\{X_i, Y_i\} \cap V(M_\wedge) \neq \emptyset$. In this case, there is a unique half-cluster F with Y_i as its dominator. Then we take (X_i, Y_i, F) as an H-triple. We assign (X_0, Y_0) as an H -pair.

Step 2. Initiating connecting edges.

Given an ε -regular pair (A, B) of density d and a subset $B' \subseteq B$, we say a vertex $a \in A$ *typical* to B' if $\deg(a, B') \geq (d - \varepsilon)|B'|$. Then by the regularity of (A, B) , the fact below holds.

Fact 4. *If (A, B) is an ε -regular pair, then at most $\varepsilon|A|$ vertices of A are not typical to $B' \subseteq B$ whenever $|B'| > \varepsilon|B|$.*

For each $1 \leq i \leq t-1$, choose $y_i^* \in Y_i$ typical to both X_i and X_{i+1} , and $y_i^{**} \in Y_i$ typical to each of X_i , X_{i+1} , and $\Gamma(y_i^*, X_i)$. Correspondingly, choose $x_{i+1}^* \in \Gamma(y_i^*, X_{i+1})$ typical to Y_{i+1} , and $x_{i+1}^{**} \in \Gamma(y_i^{**}, X_{i+1})$ typical to both Y_{i+1} and $\Gamma(x_{i+1}^*, Y_{i+1})$. For $i = t$, we choose y_t^* and y_t^{**} the same way as for $i < t$, but if in Case A, choose $x_1^* \in \Gamma(y_t^*, X_1)$ typical to Y_1 , and $x_1^{**} \in \Gamma(y_t^{**}, X_1)$ typical to both Y_1 and $\Gamma(x_1^*, Y_1)$; and if in Case B, choose $x_0^* \in \Gamma(y_t^*, X_0)$ typical to X_1 , and $x_0^{**} \in \Gamma(y_t^{**}, X_0)$ typical to both X_1 and $\Gamma(x_0^*, X_1)$. Furthermore, in Case B, we choose $y_{t+1}^* \in X_0$ typical to both Y_0 and X_1 , and $y_{t+1}^{**} \in X_0$ typical to each of Y_0 , X_1 , and $\Gamma(y_{t+1}^*, Y_0)$. Correspondingly, choose $x_1^* \in \Gamma(y_{t+1}^*, X_1)$ typical to Y_1 and $x_1^{**} \in \Gamma(y_{t+1}^{**}, X_1)$ typical to both Y_1 and $\Gamma(x_1^*, Y_1)$. Additionally, we choose $y_0^* \in \Gamma(y_{t+1}^*, Y_0)$ such that y_0^* is typical to X_0 , and choose $y_0^{**} \in \Gamma(y_{t+1}^{**}, Y_0)$ such that y_0^{**} is typical to X_0 . Notice that by the choice of these vertices above, we have the following.

$$\begin{cases} y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} \in E(G), & \text{for } 1 \leq i \leq t-1; \\ x_1^* y_t^{**}, x_1^{**} y_t^* \in E(G), & \text{in Case A;} \\ x_0^* y_t^{**}, x_0^{**} y_t^*, x_1^* y_{t+1}^*, x_1^{**} y_{t+1}^{**}, y_0^* y_{t+1}^*, y_0^{**} y_{t+1}^{**} \in E(G), & \text{in Case B.} \end{cases}$$

By Fact 4, for each $0 \leq i \leq t$, we have $|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)|, |\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)^2 N$, and $|\Gamma(y_{t+1}^*, Y_0) \cap \Gamma(y_{t+1}^{**}, Y_0)| \geq (d - \varepsilon)^2 N$.

Step 3. Super-regularizing the regular pairs in each H-pair and H-triple given in Step 1.

For each $0 \leq i \leq t$, if (X_i, Y_i) is an H -pair, let

$$X'_i = \{x \in X_i : \deg(x, Y_i) \geq (d - \varepsilon)N\} \quad \text{and} \quad Y'_i = \{y \in Y_i : \deg(y, X_i) \geq (d - \varepsilon)N\}.$$

By Fact 4, we have $|X'_i|, |Y'_i| \geq (1 - \varepsilon)N$. Recall that $x_i^*, x_i^{**} \in X_i$ and $y_i^*, y_i^{**} \in Y_i$ are the initiated vertices in Step 2. For $1 \leq i \leq t$, if $|X'_i - \{x_i^*, x_i^{**}\}| \neq |Y'_i - \{y_i^*, y_i^{**}\}|$, say $|X'_i - \{x_i^*, x_i^{**}\}| > |Y'_i - \{y_i^*, y_i^{**}\}|$, we then remove $|X'_i - \{x_i^*, x_i^{**}\}| - |Y'_i - \{y_i^*, y_i^{**}\}|$ vertices out from $X'_i - \{x_i^*, x_i^{**}\}$, and denote the remaining set still as X'_i . Denote $Y'_i - \{y_i^*, y_i^{**}\}$ still as Y'_i . We see that $|X'_i| = |Y'_i|$. As $|Y'_i| \geq (1 - \varepsilon)N$ (to be precise, the lower bound should be $(1 - \varepsilon)N - 2$, however, the constant 2 can be made vanished by adjusting the ε factor, we ignore the slight different of the ε -factor here),

we have that $|X_i \cup Y_i - (X'_i \cup Y'_i)| \leq 2\varepsilon N$. For $i = 0$, if $|X'_i - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}| \neq |Y'_i - \{y_i^*, y_i^{**}\}|$, say $|X'_i - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}| > |Y'_i - \{y_i^*, y_i^{**}\}|$, then we remove $|X'_i - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}| - |Y'_i - \{y_i^*, y_i^{**}\}|$ vertices out from $X'_i - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}$ and denote the remaining set still as X'_i . Denote $Y'_i - \{y_i^*, y_i^{**}\}$ still as Y'_i . We see that $|X'_i| = |Y'_i|$. We call the resulting H-pairs *super-regularized H-pairs*. By Slicing lemma (Lemma 3.5) and the definitions of X'_i, Y'_i , we see that

Fact 5. *Each super-regularized H-pair (X'_i, Y'_i) is a $(2\varepsilon, d - 2\varepsilon)$ -super-regular pair.*

For each H-triple (X_i, Y_i, F) , by Fact 3, (Y_i, F) is 2.1ε -regular with density at least $d - \varepsilon$. Let

$$\begin{aligned} X'_i &= \{x \in X_i : \deg(x, Y_i) \geq (d - \varepsilon)N\}, \\ Y'_i &= \{y \in Y_i : \deg(y, X_i) \geq (d - \varepsilon)N, \deg(y, F) \geq (d - 3.1\varepsilon)|F|\}, \text{ and} \\ F' &= \{f \in F : \deg(f, Y_i) \geq (d - 3.1\varepsilon)N\}. \end{aligned}$$

Recall that $x_i^*, x_i^{**} \in X_i$ and $y_i^*, y_i^{**} \in Y_i$ are the initiated vertices in Step 2. We remove x_i^*, x_i^{**} out from X'_i , and remove y_i^*, y_i^{**} out from Y'_i . Still denote the resulted clusters as X'_i and Y'_i , respectively. Remove $\lceil d^3 N \rceil$ vertices out from F , which consists of all vertices in $F - F'$ and any $\lceil d^3 N \rceil - |F - F'|$ vertices from F' (we need to increase the ratio $|Y'_i|/|X'_i \cup F'|$ a little as later on we may use vertices in Y'_i in constructing HITs covering vertices in V_0). Denote the resulting set still by F' . Then we see that $|X'_i| \geq (1 - \varepsilon)N$, $|Y'_i| \geq (1 - 3.1\varepsilon)N$, and $|F'| \geq (1 - 2.1\varepsilon)|F| - d^3 N \geq (1 - 2.1\varepsilon - 2d^3)|F|$. We call the resulted H-triples *super-regularized H-triples*. By the Slicing Lemma and the definitions above, the following is true.

Fact 6. *For each super-regularized H-triple (X'_i, Y'_i, F') , (X'_i, Y'_i) is $(2\varepsilon, d - 3.1\varepsilon)$ -super-regular, and (Y'_i, F') is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$ -super-regular.*

Let V_0^1 be the union of the set of vertices from each $(X_i \cup Y_i - (X'_i \cup Y'_i)) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} - \{y_{t+1}^*, y_{t+1}^{**}\}$ ($\{y_{t+1}^*, y_{t+1}^{**}\}$ exists only if in Case B), where (X_i, Y_i) is an H-pair, and let V_0^2 be the union of the set of vertices from each $(X_i \cup Y_i \cup F - (X'_i \cup Y'_i \cup F')) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}$, where (X_i, Y_i, F) is an H-triple. Notice that for each H-pair (X_i, Y_i) , we have $|X_i \cup Y_i - (X'_i \cup Y'_i)| \leq 2\varepsilon N$; and for each H-triple (X_i, Y_i, F) , we have $|X_i - X'_i| \leq \varepsilon N$, $|Y_i - Y'_i| \leq (\varepsilon + 2.1\varepsilon)N$, and $|F - F'| \leq d^3 N$. Hence by using the facts that $|\mathcal{W}'| \leq (1/5 - \alpha/2 + 2\beta)l$, $t = l/2$, and $Nl \leq n$ from inequality (4), we get

$$|V_0^1| + |V_0^2| \leq 2\varepsilon Nl/2 + 2(1/5 - \alpha + 2\beta)l(d^3 N + 2.1\varepsilon N) \leq 2d^3 Nl/5 + 2\varepsilon Nl \leq 2d^3 n/5 + 2\varepsilon n.$$

Let $V'_0 = V_0 \cup V_0^1 \cup V_0^2$. Then

$$|V'_0| \leq 2\varepsilon n + 2d^3 n/5 + 2\varepsilon n \leq d^3 n/2 \quad (\text{provided that } \varepsilon \leq d^3/40). \quad (5)$$

Step 4. Construct small HITs covering all vertices in V'_0 .

Consider a vertex $x \in V'_0$ and a cluster or a half-cluster A , we say that x is *adjacent to* A , denoted by $x \sim A$, if $\deg(x, A) \geq (d - \varepsilon)|A|$. We call A the *partner* of x .

Claim 4.2. *For each vertex $x \in V'_0$, there is a cluster or a half-cluster A such that $x \sim A$, where A is not a dominator, and we can assign all vertices in V'_0 to their partners which are not dominators such that each of the cluster or half-cluster is used by at most $\frac{d^2 N}{20}$ vertices from V'_0 .*

Proof. Suppose we have found partners for the first $m < d^3 n/2$ (recall that $|V'_0| \leq d^3 n/2$) vertices of V'_0 such that no cluster or half-cluster is used by at most $\frac{d^2 N}{20}$ vertices. Let Ω be the set of all clusters and half-clusters that are used exactly by $\frac{d^2 N}{20}$ vertices. Then

$$\begin{aligned} \frac{d^2 N}{20} |\Omega| &\leq m < d^3 n/2 \leq d^3 (2kN + 2\epsilon n)/2 \\ &\leq d^3 kN + d^3 \frac{2kN}{1-2\epsilon}, \end{aligned}$$

by inequality (4). Therefore,

$$\begin{aligned} |\Omega| &\leq \frac{20d^3 k}{d^2} + \frac{20d^3 l}{d^2(1-2\epsilon)} \\ &\leq 10dl + 40dl \text{ (provided that } 1-2\epsilon \geq 1/2 \text{)} \\ &\leq \beta l \text{ (provided that } 50d \leq \beta \text{)}. \end{aligned}$$

Consider now a vertex $v \in V'_0$ not having a partner found so far. Let \mathcal{U} be the set of all non-dominator clusters and half-clusters adjacent to v not contained in Ω . We claim that $|\mathcal{U}| \geq (\alpha - 7\beta)l$. To see this, we first observe that any vertex $v \in V'_0$ is adjacent to at least $(\alpha - 6\beta)l$ non-dominator clusters and half-clusters. For instead, as v may adjacent to $2|\mathcal{W}'|$ dominators, vertices in V'_0 , or clusters A with $\deg(v, A) < (d - \epsilon)|A|$, we have

$$\begin{aligned} (2/5 - \beta)n &\leq \deg_G(v) < (\alpha - 6\beta)lN + (2/5 + 4\beta - \alpha)lN + d^3 n/2 + (d - \epsilon)lN \\ &\leq (2/5 - 2\beta + d^3/2 + d - \epsilon)n \\ &< (2/5 - 3\beta/2)n \text{ (provided that } d - \epsilon + d^3/2 < \beta/2 \text{)}, \end{aligned}$$

showing a contradiction. Since $|\Omega| \leq \beta l$, we have $|\mathcal{U}| \geq (2\alpha - 7\beta)l$. \square

Now for each non-dominator cluster A (A is either a cluster X'_i , Y'_i , or a half cluster F'), let $I(A)$ be the set of vertices from V'_0 such that each of them has A as its partner. By Claim 4.2, we have $|I(A)| \leq \frac{d^2 N}{20}$.

We need three operations below for constructing small HITs covering vertices in V'_0 .

Operation I Let (A, B) be an (ϵ', δ) -super-regular pair, and I a set of vertices disjoint from $A \cup B$. Suppose that (i) $\deg(x, B) \geq d'|B| > \epsilon'|B|$ and $\deg(x, B) \geq d'|B| \geq 3|I|$ for any $x \in I$; (ii) $(\delta - \epsilon')d'|B| \geq 3|I|$; (iii) $(\delta - \epsilon')|A| > |I|$; and (iv) $\delta|A| > 4|I|$. Then we can do the following operations on (A, B) and I .

Let $I = \{x_1, x_2, \dots, x_{|I|}\}$. We first assume that $|I| \geq 2$.

Since (A, B) is (ϵ', δ) -super-regular, for each $v \in \Gamma(x_i, B)$, $|\Gamma(v, A)| \geq \delta|A|$. By condition (i), we have $|\Gamma(x_i, B)| > \epsilon'|B|$ for each i . Applying Fact 4, we then know that there are at least

$(\delta - \varepsilon')|A| > |I|$ vertices from $\Gamma(v, A)$ typical to $\Gamma(x_{i+1}, B)$ for each $1 \leq i \leq |I| - 1$. That is, there exists $A_1 \subseteq \Gamma(v, A)$ with $|A_1| \geq (\delta - \varepsilon')|A| > |I|$ such that for each $a_1 \in A_1$, $|\Gamma(a_1, \Gamma(x_{i+1}, B))| \geq (\delta - \varepsilon')d'|B| \geq 3|I|$. As $\deg(x, B) \geq d'|B| \geq 3|I|$ for any $x \in I$ and $(\delta - \varepsilon')d'|B| \geq 3|I|$, combining the above argument, we know there is a claw-matching M_I from I to B centered in I such that one vertex from $\Gamma(x_i, V(M_I))$ and one vertex from $\Gamma(x_{i+1}, V(M_I))$ have at least $(\delta - \varepsilon')|A| > |I|$ common neighbors in A . Let x_{i1}, x_{i2}, x_{i3} be the three neighbors of x_i in M_I (in fact in B) and suppose that $|\Gamma(x_{i3}, A) \cap \Gamma(x_{i+1,1}, A)| \geq |I|$. For $1 \leq i \leq |I| - 1$, we then choose distinct vertices $y_i \in \Gamma(x_{i3}, A) \cap \Gamma(x_{i+1,1}, A)$. By condition (iv), there is a \wedge -matching M_2 between the vertex set $\{x_{i3} : 1 \leq i \leq |I| - 1\}$ and the vertex set $A - \{y_i : 1 \leq i \leq |I| - 1\}$ centered in the first set, a matching M_3 between $\{x_{i+1,1} : 1 \leq i \leq |I| - 1\}$ and $A - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2)$ covering the first set, and a matching M_4 between the vertex set $\{y_i : 1 \leq i \leq |I| - 1\}$ and $B - V(M_I)$ covering the first set. Finally, by using (iv) again, we can find three distinct vertices $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A) - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2) - V(M_3)$. Let T_B be the graph with

$$V(T_B) = V(M_I) \cup \{y_i : 1 \leq i \leq |I| - 1\} \cup V(M_2) \cap V(M_3) \cup V(M_4) \cup \{y_{31}, y_{32}, y_{33}\}$$

and

$$E(T_B) = M_I \cup \{y_i x_{i3}, y_i x_{i+1,1} : 1 \leq i \leq |I| - 1\} \cup M_2 \cup M_3 \cup M_4 \cup \{x_{13} y_{31}, x_{13} y_{32}, x_{13} y_{33}\}.$$

If $|I| = 1$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, B)$ and $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A)$. Then let T_B be the graph with

$$V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}, y_{33}\}$$

and

$$E(T_B) = \{x_1 x_{11}, x_1 x_{12}, x_1 x_{13}, x_{13} y_{31}, x_{13} y_{32}, x_{13} y_{33}\}.$$

In any case, we see that T_B is a HIT satisfying

$$\begin{aligned} |V(T_B) \cap B| &= |V(T_B) \cap A| = 4|I| - 1, \\ |L(T_B) \cap B| &= \min\{2|I| + 1, 3|I| - 1\}, |L(T_B) \cap A| = 3|I|. \end{aligned} \quad (6)$$

We call T_B the insertion HIT associated with B . Figure 1 gives a depiction of T_B for $|I| = 1, 3$, respectively.

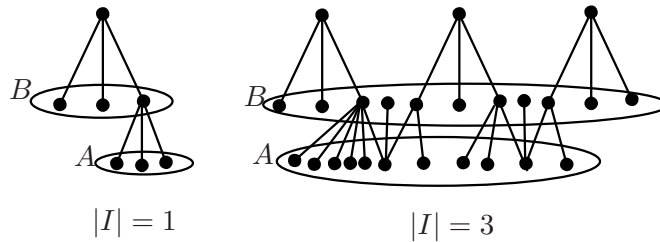


Figure 1: The HIT T_B

Operation II Let (A, B) be an (ε', δ) -super-regular pair, and I a set of vertices disjoint from $A \cup B$. Suppose that (i) $\deg(x, A) \geq d'|A| > \varepsilon'|A|$ and $\deg(x, A) \geq d'|A| \geq 3|I|$ for any $x \in I$; (ii)

$(\delta - \varepsilon')d'|A| \geq 3|I|$; (iii) $(\delta - 2\varepsilon')|B| > |I|$; and (iv) $\delta|B| > 3|I|$. Then we can do the following operations on (A, B) and I .

Let $I = \{x_1, x_2, \dots, x_{|I|}\}$. We first assume that $|I| \geq 3$.

Since (A, B) is (ε', δ) -super-regular, for each $v \in \Gamma(x_i, A)$, $|\Gamma(v, B)| \geq \delta|B|$. By condition (i), we have $|\Gamma(x_i, A)| > \varepsilon'|A|$ for each i . Applying Fact 4, we then know that there are at least $(\delta - 2\varepsilon')|B| > |I|$ vertices from $\Gamma(v, B)$ typical to both $\Gamma(x_{i+1}, A)$ and $\Gamma(x_{i+2}, A)$ for each $1 \leq i \leq |I| - 2$. That is, there exists $B_1 \subseteq \Gamma(v, B)$ with $|B_1| \geq (\delta - 2\varepsilon')|B| > |I|$ such that for each $b_1 \in B_1$, $|\Gamma(b_1, \Gamma(x_{i+1}, A))|, |\Gamma(b_1, \Gamma(x_{i+2}, A))| \geq (\delta - \varepsilon')d'|A| \geq 3|I|$. As $\deg(x, A) \geq d'|A| \geq 3|I|$ for any $x \in I$ and $(\delta - \varepsilon')d'|A| \geq 3|I|$, combining the above argument, we know there is a claw-matching M_I from I to A centered in I such that any one vertex from $\Gamma(x_i, V(M_I))$, any one vertex from $\Gamma(x_{i+1}, V(M_I))$, and any one vertex from $\Gamma(x_{i+2}, V(M_I))$ have at least $|I|$ common neighbors in B . Let x_{i1}, x_{i2}, x_{i3} be the three neighbors of x_i in M_I (in fact in A). For $i = 1$, choose $y_0 \in \Gamma(x_{13}, A) \cap \Gamma(x_{23}, A) \cap \Gamma(x_{33}, A)$. Let $h = \lceil (|I| - 3)/2 \rceil$. For $1 \leq k \leq h$, we then choose distinct vertices $y_k \in \Gamma(x_{1+2k,2}, A) \cap \Gamma(x_{2+2k,3}, A) \cap \Gamma(x_{3+2k,3}, A)$ (if $|I| = 2 + 2k$, let $\Gamma(x_{3+2k,3}, A) = A$). By condition (iv), there is a matching M between the vertex set $\{x_{i3}, x_{1+2k,2} : 1 \leq i \leq |I|, 1 \leq k \leq h\}$ and the vertex set $B - \{y_0, y_k : 1 \leq k \leq h\}$ covering the first set. If $|I|$ is even, choose $y_{31}, y_{32} \in \Gamma(x_{13}, B)$ such that they have not been chosen before; if $|I|$ is odd, choose $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, B)$ such that they have not been chosen before. Let T_A be the graph with

$$V(T_A) = \begin{cases} V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd;} \end{cases}$$

and $E(T_A)$ containing all edges in $M_I \cup M \cup \{y_0x_{13}, y_0x_{23}, y_0x_{33}\}$ and all edges in

$$\begin{cases} \{x_{1+2k,2}y_k, x_{2+2k,2}y_k, x_{3+2k,2}y_k, x_{1+2h,2}y_h, x_{2+2h,2}y_h : 1 \leq k \leq h-1\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ \{x_{1+2k,2}y_k, x_{2+2k,2}y_k, x_{3+2k,2}y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd.} \end{cases}$$

If $|I| = 1$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A)$ and $y_{31}, y_{32} \in \Gamma(x_{13}, B)$, and then let T_A be the graph with

$$V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}\} \text{ and } E(T_B) = \{x_1x_{11}, x_1x_{12}, x_1x_{13}, x_{13}y_{31}, x_{13}y_{32}\}.$$

If $|I| = 2$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A)$, $x_{11}, x_{12}, x_{13} \in \Gamma(x_2, A)$, $y \in \Gamma(x_{13}, B) \cap \Gamma(x_{21}, B)$, $y_{11}, y_{12} \in \Gamma(x_{13}, B)$, and $y_{21}, y_{22} \in \Gamma(x_{21}, B)$ such that they are all distinct, then let T_A be the graph with

$$V(T_B) = \{x_i, x_{i1}, x_{i2}, x_{i3}, y, y_{i1}, y_{i2} : i = 1, 2\} \quad \text{and}$$

$$E(T_B) = \{x_ix_{i1}, x_ix_{i2}, x_ix_{i3}, x_{13}y, x_{21}y, x_{13}y_{11}, x_{13}y_{12}, x_{21}y_{21}, x_{21}y_{22}\}.$$

We see that T_A is a tree which has a degree 2 vertex y only if $|I| = 2$ and a degree 2 vertex y_h only

if $|I| > 2$ and $|I|$ is even. In addition, T_A satisfies the following.

$$\begin{aligned}
|V(T_A) \cap A| &= 3|I| \quad \text{and} \quad |L(T_A) \cap A| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3; \end{cases} \text{ and} \\
|V(T_A) \cap B| &= \begin{cases} 2, & \text{if } |I| = 1; \\ 2|I| + 1, & \text{if } |I| \geq 2; \end{cases} \text{ and} \\
|L(T_A) \cap B| &= \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3. \end{cases} \tag{7}
\end{aligned}$$

In this case, we call T_A the insertion tree associated with A . Notice that $|L(T_A) \cap A| = |L(T_A) \cap B|$ always holds. Figure 1 gives a depiction of T_A for $|I| = 1, 2, 5, 6$, respectively.

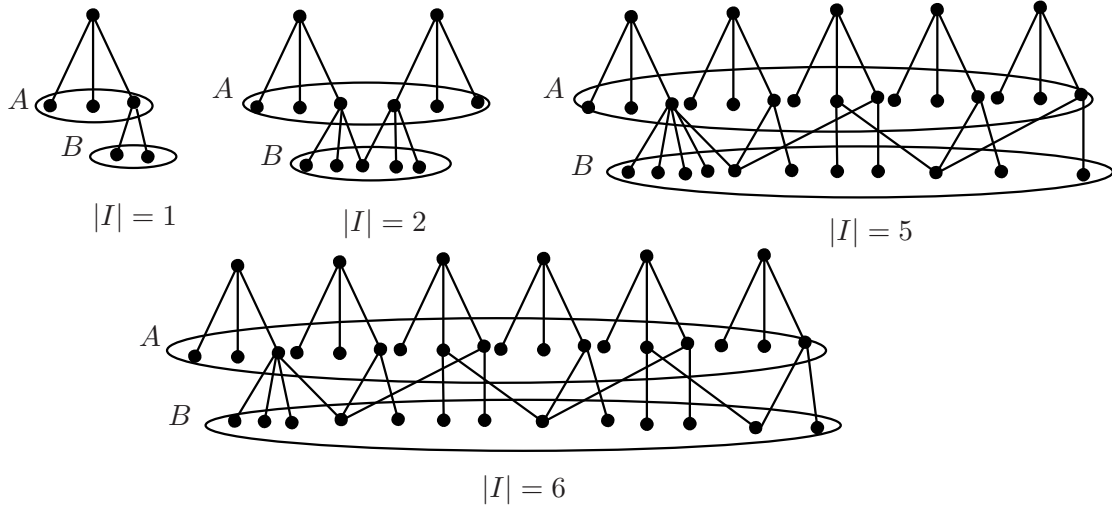


Figure 2: The tree T_A

Operation III Let (B, F) be an (ε', δ) -super-regular pair, and I a set of vertices disjoint from $B \cup F$. Suppose that $\deg(x, F) \geq d'|F| \geq 3|I|$ for any $x \in I$ and $\delta|B| \geq 6|I|$. Then we can do the following operations on (A, B) and I .

Let $I = \{x_1, x_2, \dots, x_{|I|}\}$. Since $\deg(x, B) \geq d'|B| \geq 3|I|$ for any $x \in I$, there is a claw-matching M_I from I to F centered in I . Then as $\delta|B| \geq 6|I|$, there is a \wedge -matching M_\wedge from $V(M_I) \cap F$ to B centered in $V(M_I) \cap F$. Let T_F be the graph with

$$V(T_F) = V(M_I) \cup V(M_\wedge) \quad \text{and} \quad E(T_F) = M_I \cup M_\wedge.$$

We see that T_F is a forest with no vertex of degree 2 satisfying

$$|V(T_F) \cap F| = |S(T_F) \cap F| = 3|I| \quad \text{and} \quad |V(T_F) \cap B| = |L(T_F) \cap B| = 6|I|. \tag{8}$$

We call T_F the insertion forest associated with F .

Now for each H-pair (X'_i, Y'_i) , we may assume that $I(X'_i) \neq \emptyset$ and $I(Y'_i) \neq \emptyset$ for a uniform discussion, as the consequent argument is independent of the assumptions. Recall that (X'_i, Y'_i) is $(2\varepsilon, d - 2\varepsilon)$ -super-regular by Fact 5. Notice that $\deg(x, X'_i) \geq (d - \varepsilon)|X'_i|$ for each $x \in I(X'_i)$, $|I(X'_i)| \leq \frac{d^2 N}{20}$, and $|X'_i|, |Y'_i| \geq (1 - \varepsilon)N$. By simple calculations, we see that (i) $\deg(x, X'_i) \geq (d - \varepsilon)|X'_i| > 2\varepsilon|X'_i|$ and $(d - \varepsilon)|X'_i| \geq 3d^2 N/20$ for each $x \in I(X'_i)$; (ii) $(d - 2\varepsilon - 2\varepsilon)(d - \varepsilon)|X'_i| > 3d^2 N/20$; (iii) $(d - 4\varepsilon)|Y'_i| > d^2 N/20$; and (iv) $(d - 2\varepsilon)|Y'_i| > d^2 N/5 \geq 4I(X'_i)$. Thus all the conditions in Operation I are satisfied. So we can find a HIT $T_{X'_i}$ associated with X'_i . As $|V(T_{X'_i}) \cap X'_i| = |V(T_{X'_i}) \cap Y'_i| \leq 4|I(X'_i)| \leq \frac{d^2 N}{5}$, we know that $(X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))$ is $(4\varepsilon, d - 2\varepsilon - d^2 N/5)$ -super regular. Since $\deg(y, Y'_i) \geq (d - \varepsilon)|Y'_i|$ for each $y \in I(Y'_i)$, we get $\deg(y, Y'_i - V(T_{X'_i})) \geq (d - \varepsilon - d^2/5)|Y'_i|$ for each $y \in I(Y'_i)$. By direct checking, conditions (i) \sim (iv) of Operation I are satisfied by the pair $(X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))$ and $I(Y'_i)$. Then we use Operation I on $(X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))$ and $I(Y'_i)$ to get a HIT $T_{Y'_i}$ associated with $Y'_i - V(T_{X'_i})$. Denote

$$X_i^* = X'_i - V(T_{X'_i}) - V(T_{Y'_i}) \quad \text{and} \quad Y_i^* = Y'_i - V(T_{X'_i}) - V(T_{Y'_i}).$$

By using (6) in Operation I, we have $|X_i^*| = |Y_i^*| \geq (1 - 2d^2/5 - \varepsilon)N \geq N/2$. By Slicing lemma (Lemma 3.5) and Fact 5, we have the following.

Fact 7. *For each H-pair (X_i, Y_i) , (X_i^*, Y_i^*) is $(4\varepsilon, d - 2\varepsilon - 2d^2/5)$ -super-regular with $|X_i^*| = |Y_i^*|$. We call (X_i^*, Y_i^*) a ready H-pair.*

Then for each H-triple (X'_i, Y'_i, F') , we may assume that $I(X'_i) \neq \emptyset$ and $I(F') \neq \emptyset$ (recall that Y_i is assumed to be the dominator of F , so $I(Y'_i) = \emptyset$ by the distribution principle of vertices in V'_0 from Claim 4.2). By Fact 6, we know that (X'_i, Y'_i) is $(2\varepsilon, d - 3.1\varepsilon)$ -super-regular and (Y'_i, F') is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$ -super-regular. Notice also that $|X'_i| \geq (1 - \varepsilon)N$, $|Y'_i| \geq (1 - 3.1\varepsilon)N$, $|F'| \geq (1 - 2.1\varepsilon - 2d^3)N/2$, and $\deg(x, X'_i) \geq (d - \varepsilon)|X'_i|$ and $\deg(y, F') \geq (d - \varepsilon)|F'|$ for each $x \in I(X'_i)$ and each $y \in I(F')$. Since $|I(X'_i)|, |I(F')| \leq \frac{d^2 N}{20}$ and $\varepsilon \ll d \ll 1$, the conditions of Operation III are satisfied by (Y'_i, F') and $I(F')$ by direct calculations. Let $T_{F'}$ be the insertion forest associated with F' . Then we use Operation II on $(X'_i, Y'_i - V(T_{F'}))$ and $I(X'_i)$ to get a tree $T_{X'_i}$ associated with X'_i . Denote

$$X_i^* = X'_i - V(T_{X'_i}), \quad Y_i^* = Y'_i - V(T_{F'}) - V(T_{X'_i}), \quad \text{and} \quad F^* = F' - V(T_{F'}).$$

By using (7) and (8) in Operation II and Operation III, respectively, we have $|X_i^*|, |Y_i^*| \geq (1 - 3.1\varepsilon - 9d^2/20)N \geq N/2$ and $|F^*| \geq (1 - 2.1\varepsilon - 2d^3)N/2 - 3d^2 N/20 \geq (1 - 2.1\varepsilon - 2d^3 - 3d^2/10)N/2$. By Slicing lemma and Fact 6, we have the following.

Fact 8. *For each H-triple (X_i, Y_i, F) , (X_i^*, Y_i^*) is $(4\varepsilon, d - 3.1\varepsilon - 9d^2/20)$ -super-regular and (Y_i^*, F^*) is $(8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)$ -super-regular. We call (X_i^*, Y_i^*, F^*) a ready H-triple.*

Step 5. Apply the Blow-up lemma to find a HIT within each ready H-pair and among each ready H-triple.

In order to apply the Blow-up Lemma, we first give two lemmas which assure the existence of a given subgraph in a complete bipartite graph.

Lemma 4.4. *Suppose $0 < \varepsilon \ll d \ll 1$ and N is a large integer. If $G(A, B)$ is a balanced complete bipartite graph with $(1 - \varepsilon - d^2/2)N \leq |A| = |B| \leq N$, then $G(A, B)$ contains a HIST T_{pair} with $\Delta(T_{\text{pair}}) \leq \lceil 2/d^3 \rceil$ and $||L(T_{\text{pair}}) \cap A| - |L(T_{\text{pair}}) \cap B|| = \ell$ for any given non-negative integer ℓ with $\ell \leq d^2 N$.*

Proof. By the symmetry, we only show that we can construct a HIST T such that $|L(T) \cap A| - |L(T) \cap B| = \ell$. Let $\Delta' = \lceil d^3 N \rceil$. We choose distinct $a_1, a_2, \dots, a_{\Delta'} \in A$ and distinct $b_1, b_2, \dots, b_{\Delta'-1} \in B$. Then we decompose all vertices in B into $B_1, B_2, \dots, B_{\Delta'}$ such that $3 \leq |B_i| \leq 1/d^3$, $B_i \cap B_{i+1} = \{b_i\}$ for $1 \leq i \leq \Delta' - 1$, and $B_i \cap B_j = \emptyset$ for $|i - j| > 1$. Now we choose $\ell + 1$ distinct vertices $b_{\Delta'}, b_{\Delta'+1}, \dots, b_{\Delta'+\ell}$ from $B - \{b_i : 1 \leq i \leq \Delta' - 1\}$. As $\Delta' = \lceil d^3 N \rceil$, $\ell + \Delta' \leq (d^2 + d^3)N + 1$, and thus

$$2(\ell + \Delta') \leq (2d^2 + 2d^3)N + 2 \leq (1 - d^2/2 - \varepsilon)N - \lceil d^3 N \rceil \leq |A| - \lceil d^3 N \rceil.$$

Thus we can use all of the vertices in $\{b_i : 1 \leq i \leq \Delta' + \ell\}$ to cover all vertices in $A - \{a_i | 1 \leq i \leq \Delta' - 1\}$ such that each b_i can be adjacent to at least two distinct vertices. We partition $A - \{a_i | 1 \leq i \leq \Delta' - 1\}$ arbitrarily into $A_1, A_2, \dots, A_{\ell+\Delta'}$ such that $2 \leq |A_i| \leq 1/d^3$. Now let T be a spanning subgraph of $G(A, B)$ such that

$$E(T) = \{a_i b | b \in B_i, 1 \leq i \leq \Delta'\} \cup \{b_j a | a \in A_j, 1 \leq j \leq \Delta' + \ell\}.$$

Clearly, $\Delta(T) \leq \lceil 2/d^3 \rceil$. As $|A| = |B|$, $|S(T) \cap A| = \Delta'$, and $|S(T) \cap B| = \Delta' + \ell$, we then have that $|L(T) \cap A| - |L(T) \cap B| = \ell$. We denote T as T_{pair} . \blacksquare

Lemma 4.5. *Suppose $0 < \varepsilon \ll d \ll 1$ and N is a large integer. Let $G = G(A, B, F)$ be a tripartite graph with $V(G)$ partitioned into $A \cup B \cup F$ such that both $G[A \cup B]$ and $G[B \cup F]$ are complete bipartite graphs. If (i) $(1 - 4\varepsilon - d^2/2)N \leq |A|, |B| \leq N$, (ii) $(1/2 - 2.1\varepsilon - 3d^2/20 - d^3)N \leq |F| \leq (1/2 - d^3)N$, and (iii) for any given non-negative integer $l \leq 3d^2 N/10$, we have $|B| - 2(|A \cup F| - |B| - l) \geq 3d^3 N/2$ holds, then G contains a HIST T_{triple} and a path P_{triple} spanning on a subset of $L(T_{\text{triple}})$ such that*

- (a) T_{triple} is a HIST of G with $\Delta(T_{\text{triple}}) \leq \lceil 3/d^3 \rceil$;
- (b) $|L(T_{\text{triple}}) \cap B| = |L(T_{\text{triple}}) \cap (A \cup F)| - l$.
- (c) P_{triple} is a (b, f) -path on $L(T_{\text{triple}}) \cap F$ and any $|L(T_{\text{triple}}) \cap F|$ vertices from $L(T_{\text{triple}}) \cap B$, and $|V(P_{\text{triple}}) \cap F| \leq 5d^2 N/6$.

Proof. Let $\Delta' = \lceil d^3 N/2 \rceil$. We choose distinct $b_1, b_2, \dots, b_{\Delta'} \in B$ and partition all vertices in F into $F_1, F_2, \dots, F_{\Delta'}$ such that $3 \leq |F_i| \leq 1/d^3$. Then we choose distinct $a_1, a_2, \dots, a_{\Delta'-1} \in A$ and decompose all vertices in A into $A_1, A_2, \dots, A_{\Delta'}$ such that $3 \leq |A_i| \leq 2/d^3$, $A_i \cap A_{i+1} = \{a_i\}$ for $1 \leq i \leq \Delta' - 1$, and $A_i \cap A_j = \emptyset$ for $|i - j| > 1$. Choose one more vertex, say $a_{\Delta'} \in A - \{a_i | 1 \leq$

$i \leq \Delta' - 1\}$. Let $l' = |A \cup F| - |B| - l$. Notice that $l' > 0$. Now we choose l' distinct vertices $f_1, f_2, \dots, f_{l'}$ from $A - \{a_i : 1 \leq i \leq \Delta'\} \cup F$ (choose as many as possible from F first) and partition any $2l'$ vertices of $B - \{b_i : 1 \leq i \leq \Delta'\}$ into $B_1, B_2, \dots, B_{l'}$ such that $|B_i| = 2$. By (iii), we see that there are at least $\lfloor d^3 N \rfloor$ vertices left in $B' = B - \{b_i : 1 \leq i \leq \Delta'\} - \bigcup_{i=1}^{l'} \{B_i\}$. Hence we can partition $B' = B'_1 \cup B'_2 \cup \dots \cup B'_{\Delta'}$ such that $|B'_{\Delta'}| \geq 2$ and $|B'_j| \geq 1$ for $j \neq \Delta'$. We let T be a subgraph of G on $A \cup B \cup F$ with

$$E(T) = \{b_i f, b_i a, a_i b' : f \in F_i, a \in A_i, b' \in B'_i, 1 \leq i \leq \Delta'\} \cup \{f_i b : b \in B_i, 1 \leq i \leq l'\}.$$

By the construction, T is a HIST of G , which clearly satisfies (a). Since $|S(T) \cap B| = \Delta'$ and $|S(T) \cap (A \cup F)| = \Delta' + l' = \Delta' + |A \cup F| - |B| - l$, we then see that T satisfies (b). If $L(T) \cap F \neq \emptyset$, let $f \in L(T) \cap F$ and $b \in L(T) \cap B$, we can then take a (b, f) -path P with $V(P) \cap F = L(T) \cup F$ and $|V(P)| = 2|L(T) \cap F|$. By (i) and (ii), we see that $l' = |A \cup F| - |B| - l \geq (1/2 - 6.1\varepsilon - 4d^2/5 - d^3)N$. Hence $|V(P) \cap F| = |F| - l' \leq 5d^2 N/6$. Denote T as T_{triple} and P as P_{triple} . \blacksquare

Now for $1 \leq i \leq t$ and for each ready H-pair (X_i^*, Y_i^*) , suppose, without of loss generality, that $|(L(T_{X'_i}) \cap Y'_i) \cup ((L(T_{Y'_i}) \cap Y'_i)| - |((L(T_{X'_i}) \cap X'_i) \cup ((L(T_{Y'_i}) \cap X'_i)| = l'$, where $T_{X'_i}$ is the insertion HIT associated with X'_i and $T_{Y'_i}$ is the insertion HIT associated with Y'_i . Notice that $l' \leq d^2 N$ from (6) and (7). Let $x_a \in S(T_{X'_i}) \cap X'_i$ be a non-leaf of $T_{X'_i}$ and $y_b \in S(T_{Y'_i}) \cap Y'_i$ a non-leaf of $T_{Y'_i}$. Since (X'_i, Y'_i) is $(2\varepsilon, d - 2\varepsilon)$ -super-regular by Fact 5 and $|Y'_i - Y_i^*| \leq 2d^2 N/5$, we have $\deg(x_a, Y_i^*) \geq (d - 2\varepsilon - d^2/2)N \geq dN/2$. Similarly, $\deg(y_b, X_i^*) \geq (d - 2\varepsilon - d^2/2)N \geq dN/2$. Also, from Step 2, we have $\Gamma(x_i^*, Y_i), \Gamma(x_i^{**}, Y_i) \geq (d - 3\varepsilon)N$. So, $\Gamma(x_i^*, Y_i^*), \Gamma(x_i^{**}, Y_i^*) \geq (d - 3\varepsilon - d^2/2)N \geq dN/2$. Similarly, we have $\Gamma(y_i^*, X_i^*), \Gamma(y_i^{**}, X_i^*) \geq (d - 3\varepsilon - d^2/2)N \geq dN/2$. Recall that (X_i^*, Y_i^*) is $(4\varepsilon, d - 2\varepsilon - 8d^2/20)$ -super-regular by Fact 7, and therefore (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$ -super-regular. By the the strengthened version of the Blow-up lemma and Lemma 4.4 (the conditions are certainly satisfied by X_i^* and Y_i^*), we can find a HIST $T_1^i \cong T_{\text{pair}}$ on $X_i^* \cup Y_i^*$ such that there exist $y_a \in S(T_1^i) \cap \Gamma(x_a, Y_i^*)$, $x_b \in S(T_1^i) \cap \Gamma(y_b, X_i^*)$, $y'_i \in S(T_1^i) \cap \Gamma(x_i^*, Y_i)$, $y''_i \in S(T_1^i) \cap \Gamma(x_i^{**}, Y_i)$, and $x'_i \in S(T_1^i) \cap \Gamma(y_i^*, X_i)$, $x''_i \in S(T_1^i) \cap \Gamma(y_i^{**}, X_i)$ such that $|L(T_1^i) \cap X_i^*| - |L(T_1^i) \cap Y_i^*| = l'$. Hence $|L(T_1^i) \cap X_i^*| + |L(T_{X'_i}) \cap X'_i| + |L(T_{Y'_i}) \cap Y'_i| = |L(T_1^i) \cap Y_i^*| + |L(T_{X'_i}) \cap Y'_i| + |L(T_{Y'_i}) \cap Y'_i|$. Let $T^i = T_1^i \cup T_{X'_i} \cup T_{Y'_i} \cup \{x_a y_a, y_b x_b\} \cup \{x_i^* y'_i, x_i^{**} y''_i, y_i^* x'_i, y_i^{**} x''_i\}$. It is clear that T^i is a HIST on $X'_i \cup Y'_i \cup I(X'_i) \cup I(Y'_i)$ such that

$$\{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X'_i| = |L(T^i) \cap Y'_i|.$$

For the ready H-pair (X_0^*, Y_0^*) , let $x_a \in S(T_{X'_0}) \cap X'_0$ be a non-leaf of $T_{X'_0}$ and $y_b \in S(T_{Y'_0}) \cap Y'_0$ a non-leaf of $T_{Y'_0}$. By the the strengthened version of the Blow-up lemma and Lemma 4.4 (the conditions are certainly satisfied by X_0^* and Y_0^*), we can find a HIST $T_1^0 \cong T_{\text{pair}}$ on $X_0^* \cup Y_0^*$ such that there exist $y'_0 \in S(T_1^0) \cap \Gamma(x_0^*, Y_0)$, $y''_0 \in S(T_1^0) \cap \Gamma(x_0^{**}, Y_0)$, $x'_{t+1} \in S(T_1^0) \cap \Gamma(y_{t+1}^*, Y_0)$, $x''_{t+1} \in S(T_1^0) \cap \Gamma(y_{t+1}^{**}, Y_0)$, and $x'_0 \in S(T_1^0) \cap \Gamma(y_0^*, X_0)$, $x''_0 \in S(T_1^0) \cap \Gamma(y_0^{**}, X_0)$ such that $|L(T_1^0) \cap X_0^*| + |L(T_{X'_0}) \cap X'_0| + |L(T_{Y'_0}) \cap Y'_0| = |L(T_1^0) \cap Y_0^*| + |L(T_{X'_0}) \cap Y'_0| + |L(T_{Y'_0}) \cap Y'_0| + 2$. Let $T^0 = T_1^0 \cup T_{X'_0} \cup T_{Y'_0} \cup \{x_a y_a, y_b x_b\} \cup \{x_0^* y'_0, x_0^{**} y''_0, y_0^* x'_0, y_0^{**} x''_0, y_{t+1}^* x'_{t+1}, y_{t+1}^{**} x''_{t+1}\}$. It is clear that T^0 is a HIST on $X'_0 \cup Y'_0 \cup I(X'_0) \cup I(Y'_0)$ such that

$$\{x_0^*, x_0^{**}, y_0^*, y_0^{**}, y_{t+1}^*, y_{t+1}^{**}\} \subseteq L(T^0) \quad \text{and} \quad |L(T^0) \cap X'_0| = |L(T^0) \cap Y'_0| + 2.$$

For each ready triple (X_i^*, Y_i^*, F^*) , we know that (X_i^*, Y_i^*) is $(4\varepsilon, d - 3.1\varepsilon - 9d^2/20)$ -super-regular and (Y_i^*, F^*) is $(8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)$ -super-regular by Fact 8. Notice that $(1 - 4\varepsilon - 9d^2/20)N \leq |X_i^*|, |Y_i^*| \leq N$ and $(1/2 - 2.1\varepsilon - 3d^2/30 - d^3)N \leq |F^*| \leq (1/2 - d^3)N$. Let $|I(X_i')| = l'$ and $|I(F')| = l/6$ for some integer l . By Operation II we have $|V(T_{X_i'}) \cap X_i'| \leq 3l'$ and $|V(T_{X_i'}) \cap Y_i'| \leq 2l' + 1$. By Operation III we have $|V(T_{F'}) \cap F_i'| = l/2$ and $|V(T_{F'}) \cap Y_i'| = l$. Notice that $|L(T_{X_i'}) \cap X_i'| = |L(T_{X_i'}) \cap Y_i'|$. Hence,

$$\begin{aligned} |Y_i^*| - 2(|X_i^* \cup F^*| - |Y_i^*| - l) &\geq 3(|Y_i'| - 2l' - l - 1) - 2(|X_i'| - 3l') - 2(|F'| - l/2) + 2l \\ &= 3|Y_i'| - 2|X_i'| - 2|F'| - 3 \\ &\geq 3(1 - 3.1\varepsilon)N - 2N - N + 2d^3N - 3 > 3d^3N/2. \end{aligned}$$

By the weak version of the Blow-up lemma (Lemma 3.2) and Lemma 4.5, we then can find a HIT $T_1^i \cong T_{\text{triple}}$ on $X_i^* \cup Y_i^* \cup F^*$ and a path $P_i \cong P_{\text{triple}}$ spanning on $L(T_1^i) \cap F^*$ and other $|L(T_1^i) \cap F^*|$ vertices from Y_i^* . Let $y_a \in S(T_{X_i'}) \cap Y_i'$ be a non-leaf of $T_{X_i'}$ (take y_a as the degree 2 vertex if $T_{X_i'}$ has one) and $y'_a \in S(T_{F'}) \cap Y_i'$ a non-leaf of $T_{F'}$. Then as (Y_i', F') is $(4.1\varepsilon, d - 2.1\varepsilon - 2d^3)$ -super-regular, we have $|\Gamma(y_a, F')|, |\Gamma(y'_a, F')| \geq (d - 2.1\varepsilon - 2d^3)N/2$. Since $|F' - F^*| \leq 3d^2N/20$, we then know that $|\Gamma(y_a, F^*)|, |\Gamma(y'_a, F^*)| \geq (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2$. Since $|F^* \cap L(T_1^i)| = |V(P_i) \cap F^*| \leq 5d^2N/6 < (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2$, there exist $f_a \in (S(T_1^i) \cap F^*) \cap \Gamma(y_a, F^*)$ and $f'_a \in (S(T_1^i) \cap F^*) \cap \Gamma(y'_a, F^*)$. For each $x \in I(F')$, since $\deg(x, F') \geq (d - \varepsilon)|F'| \geq (d - \varepsilon)(1 - 2.1\varepsilon - d^3)N/2$, we know there exists $f' \in (S(T_1^i) \cap F^*) \cap \Gamma(x, F^*)$. From Step 2, we have $|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)| \geq (d - \varepsilon)^2N$ and $|\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)^2N$. Hence $|\Gamma(x_i^*, Y_i') \cap \Gamma(x_i^{**}, Y_i')| \geq ((d - \varepsilon)^2 - 3.1\varepsilon)N$. Since $|S(T_1^i \cup T_{X_i'} \cup T_{F'}) \cap X_i'| < d^2N/2$, we see that there exists $y' \in \Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i) \cap L(T_1^i \cup T_{X_i'} \cup T_{F'})$. Similarly, there exists $x' \in \Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i) \cap L(T_1^i \cup T_{X_i'} \cup T_{F'})$. Let $T^i = T_1^i \cup T_{X_i'} \cup T_{F'} \cup \{xf' : x \in I(F'), f' \in (S(T_1^i) \cap F^*) \cap \Gamma(x, F^*)\} \cup \{y_af_a, y'_af'_a\} \cup \{y'x_i^*, y'x_i^{**}, x'y_i^*, x'y_i^{**}\}$. It is clear that T^i is a HIST on $X_i' \cup Y_i' \cup F' \cup I(X_i') \cup I(F')$ such that

$$\{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X_i'| = |L(T^i) \cap Y_i'|.$$

Let $H^i = T^i \cup P_i$. We call P_i the accompany path of T^i .

Step 6. Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two vertex-disjoint paths covering all the leaves. Then connect all the HITs into a HIST of G and connect the disjoint paths into a cycle using the edges initiated in Step 2.

Suppose $1 \leq i \leq t$. For each H-pair (X_i, Y_i) , let $X_i^L = X_i' \cap L(T^i) - \{x_i^*, x_i^{**}\}$ and $Y_i^L = Y_i' \cap L(T^i) - \{y_i^*, y_i^{**}\}$, and for each H-triple (X_i, Y_i, F) , let $X_i^L = X_i' \cap L(T^i \cup P_i) - \{x_i^*, x_i^{**}\}$ and $Y_i^L = Y_i' \cap L(T^i \cup P_i) - \{y_i^*, y_i^{**}\}$, where T^i is the HIST found in Step 5, and P_i is the accompany path of T^i . By Operations I, II and III, and the proofs of the Lemmas 4.4 and 4.5, we have $I(X_i') \cup I(Y_i') \subseteq S(T_i)$ and $F' \cup I(F') \subseteq S(T_i \cup P_i)$. Thus, $X_i^L \cup Y_i^L = L(T^i) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}$ for each H-pair and $X_i^L \cup Y_i^L = L(T^i \cup P_i) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}$ for each H-triple. Furthermore, we have $|X_i^L| = |Y_i^L|$. For the H-pair (X_0, Y_0) , let $X_0^L = X_0' \cap L(T^0) - \{x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\}$ and $Y_0^L = Y_0' \cap L(T^0) - \{y_0^*, y_0^{**}\}$. We have $X_0^L \cup Y_0^L = L(T^0) - \{x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\}$ and $|X_0^L| = |Y_0^L|$.

since from Step 5 we have $|L(T^0) \cap X'_0| = |L(T^0) \cap Y'_0| + 2$. By the construction of T_{pair} and H_{triple} , we see that $|S(T_i) \cap X'_i|, |S(T_i) \cap Y'_i| \leq d^2 N$. Since each H-pair (X'_i, Y'_i) is $(2\varepsilon, d - 2\varepsilon)$ -super-regular, and each pair (X'_i, Y'_i) from an H-triple (X'_i, Y'_i, F') is $(2\varepsilon, d - 3.1\varepsilon)$ -super-regular, by Slicing Lemma, we then know that (X_i^L, Y_i^L) is $(4\varepsilon, d - 3.1\varepsilon - d^2)$ -super-regular and hence is $(4\varepsilon, d/2)$ -super-regular.

For each $1 \leq i \leq t$, by the choice of $x_i^*, x_i^{**}, y_i^*, y_i^{**}$, we have $|\Gamma(x_i^*, Y_i)|, |\Gamma(x_i^{**}, Y_i)| \geq (d - \varepsilon)N$ and $|\Gamma(y_i^*, X_i)|, |\Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)N$. Hence, $|\Gamma(x_i^*, Y_i^L)|, |\Gamma(x_i^{**}, Y_i^L)| \geq (d - \varepsilon - 3.1\varepsilon - d^2)N > dN/2$ and $|\Gamma(y_i^*, X_i^L)|, |\Gamma(y_i^{**}, X_i^L)| \geq (d - \varepsilon - 3.1\varepsilon - d^2)N > dN/2$. Similar results hold for the vertices $x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}$. For each $0 \leq i \leq t$, we choose distinct vertices $y'_i \in \Gamma(x_i^*, Y_i^L)$, $y''_i \in \Gamma(x_i^{**}, Y_i^L)$ and $x'_i \in \Gamma(y_i^*, X_i^L)$, $x''_i \in \Gamma(y_i^{**}, X_i^L)$. If T^i does not have the accompany path, then by the strengthened version of the Blow-up lemma, we can find an (x'_i, y'_i) -path P_1^i and an (x''_i, y''_i) -path P_2^i such that $P_1^i \cup P_2^i$ is spanning on $X_i^L \cup Y_i^L$. If T^i has the accompany (b, f) -path P_i , we see that $\deg(b, X_i^L), \deg(f, Y_i^L) \geq dN/2$ as (X'_i, Y'_i) is $(2\varepsilon, d - 3.1\varepsilon)$ -super-regular, and (Y'_i, F') is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$ -super-regular. Applying the strengthened version of the Blow-up lemma, we can find an (x'_i, y'_i) -path P_{11}^i and an (x''_i, y''_i) -path P_{12}^i such that $P_{11}^i \cup P_{12}^i$ is spanning on $X_i^L \cup Y_i^L$, and two consecutive internal vertices $a', b' \in V(P_{11}^i)$ with $b' \in \Gamma(f, Y_i^L)$, and $a' \in \Gamma(b, X_i^L)$. Let $P_1^i = P_{11}^i \cup P_i \cup \{fb', ba'\} - \{a'b'\}$. Notice that for the H-pair (X_0, Y_0) , the two vertices y_{t+1}^*, y_{t+1}^{**} are not used in this step, but we will connect them to y_0^* and y_0^{**} , respectively, in next step.

We now connect the small HITs and paths together to find an SGHG of G . In Case A, for $1 \leq i \leq t - 1$, we have $|S(T^i) \cap Y_i| \geq d^3 N/2 > \varepsilon N$ and $|S(T^{i+1}) \cap X_{i+1}| \geq d^3 N/2 > \varepsilon N$. Since (Y_i, X_{i+1}) is an ε -regular pair with density d , we see that there is an edge e_i connecting $S(T^{i+1}) \cap X_{i+1}$ and $S(T^i) \cap Y_i$. Let

$$T = \bigcup_{i=1}^t T^i \cup \{e_i \mid 1 \leq i \leq t - 1\}.$$

Then T is a HIST of G . Let C be the cycle formed by all the paths in $\bigcup_{i=1}^t (P_1^i \cup P_2^i)$ and all edges in the following set

$$\{x_i^* y'_i, x_i^{**} y''_i, y_i^* x'_i, y_i^{**} x''_i : 1 \leq i \leq t\} \cup \{y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} : 1 \leq i \leq t - 1\} \cup \{y_t^* x_1^{**}, y_t^{**} x_1^*\},$$

notices that the edges in $\{y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} : 1 \leq i \leq t - 1\} \cup \{y_t^* x_1^{**}, y_t^{**} x_1^*\}$ above are guaranteed in Step 2. It is easy to see that C is a cycle on $L(T)$. Hence $H = T \cup C$ is an SGHG of G .

In Case B, for $1 \leq i \leq t - 1$, we have $|S(T^i) \cap Y_i| \geq d^3 N/2 > \varepsilon N$ and $|S(T^{i+1}) \cap X_{i+1}| \geq d^3 N/2 > \varepsilon N$. Since (Y_i, X_{i+1}) is an ε -regular pair with density d , we see that there is an edge e_i connecting $S(T^{i+1}) \cap X_{i+1}$ and $S(T^i) \cap Y_i$. Similarly, there is an edge e_0 connecting $S(T_0) \cap X_0$ and $S(T^1) \cap X_1$. Let

$$T = \bigcup_{i=1}^t T^i \cup \{e_i \mid 0 \leq i \leq t - 1\}.$$

Then T is a HIST of G . Let C be the cycle formed by all paths in $\bigcup_{i=1}^t (P_1^i \cup P_2^i)$ and all edges in

the set $\{y_0^* y_{t+1}^*, y_0^{**} y_{t+1}^{**}, y_{t+1}^* x_1^*, y_{t+1}^{**} x_1^{**}, x_0^* y_t^*, x_0^{**} y_t^*\}$ and in the following set

$$\{x_i^* y_i', x_i^{**} y_i'', y_i^* x_i', y_i^{**} x_i'' : 0 \leq i \leq t\} \cup \{y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} : 1 \leq i \leq t-1\}.$$

It is easy to see that C is a cycle on $L(T)$. Hence $H = T \cup C$ is an SGHG of G .

The proof of Theorem 4.1 is now finished. ■

4.2 Proof of Theorem 4.2

By the assumption that $\deg(v_1, V_2) \leq 2\beta n$ for each $v_1 \in V_1$ and the assumption that $\delta(G) \geq (2n+3)/5$ in Extremal Case 1, we see that

$$\delta(G[V_1]) \geq (2n+3)/5 - 2\beta n. \quad (9)$$

Then (9) implies that

$$|V_1| \geq (2n+3)/5 - 2\beta n \quad \text{and} \quad |V_2| \leq 3n/5 + 2\beta n. \quad (10)$$

Also, by $|V_2| \geq (2/5 - 4\beta)n$ in the assumption,

$$|V_1| \leq (3/5 + 4\beta)n. \quad (11)$$

We will construct an SGHG of G following several steps below.

Step 1. Repartitioning

Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. Let

$$V_1' = V_1 \quad \text{and} \quad V_2' = \{v \in V_2 \mid \deg(v, V_1) \leq \alpha_1 |V_1|\}.$$

Then by $d(V_1, V_2) \leq \alpha$, we have

$$\alpha_1 |V_1| |V_2 - V_2'| \leq e(V_1, V_2') + e(V_1, V_2 - V_2') = e(V_1, V_2) \leq \alpha |V_1| |V_2|.$$

This gives that

$$|V_2 - V_2'| \leq \alpha_2 |V_2|. \quad (12)$$

Denote $V_{12}^0 = V_2 - V_2'$. Then by the definition of V_2' , we have

$$\delta(V_{12}^0, V_1') > \alpha_1 |V_1'| \quad \text{and} \quad \delta(G[V_2']) \geq (2n+3)/5 - \alpha_1 |V_1'| \geq (2/5 - \alpha_1(3/5 + 4\beta))n, \quad (13)$$

where the last inequality follows from (11).

Let $n_i = |V_i'|$ for $i = 1, 2$. Then by (9) and (11),

$$\delta(G[V_1']) \geq (2n+3)/5 - 2\beta n \geq \frac{2/5 - 2\beta}{3/5 + 4\beta} n_1 \geq (2/3 - 8\beta)n_1, \quad (14)$$

and by (10) and the second inequality in (13),

$$\delta(G[V'_2]) \geq (2/5 - \alpha_1(3/5 + 4\beta))n \geq \frac{(2/5 - \alpha_1(3/5 + 4\beta))}{3/5 + 2\beta}n_2 \geq (2/3 - 1.1\alpha_1)n_2,$$

provided that $\beta \leq \frac{0.3\alpha_1}{9\alpha_1 + 20/3}$.

Step 2. Finding three connecting edges

As G is 3-connected, there are 3 independent edges $x_L^1 y_L^1, x_L^2 y_L^2$ and $x_N y_N$ connecting $V'_1 \cup V_{12}^0$ and V'_2 such that $x_L^1, x_L^2, x_N \in V'_1 \cup V_{12}^0$ and $y_L^1, y_L^2, y_N \in V'_2$. In the remaining steps, we will find a HIST T_1 in $G[V'_1 \cup V_{12}^0]$ with x_N as a non-leaf and x_L^1, x_L^2 as leaves, and a HIST T_2 of $G[V'_2]$ with y_N as a non-leaf and y_L^1, y_L^2 as leaves. Then $T = T_1 \cup T_2 \cup \{x_N y_N\}$ is a HIST of G . By finding a hamiltonian (x_L^1, x_L^2) -path P_1 on $L(T_1)$, and a hamiltonian (y_L^1, y_L^2) -path on $L(T_2)$, we see that

$$C := P_1 \cup P_2 \cup \{x_L^1 y_L^1, x_L^2 y_L^2\}$$

forms a cycle on $L(T)$. Hence $H := T \cup C$ is an SGHG of G .

Step 3. Initiating two HITs

In this step, we first initiate a HIT in $G[V'_1 \cup V_{12}^0]$ containing x_N as a non-leaf and x_L^1 and x_L^2 as leaves. Then, we initiate a HIT in $G[V'_2]$ containing y_N as a non-leaf and y_L^1 and y_L^2 as leaves.

For $x_L^1, x_L^2, x_N \in V'_1 \cup V_{12}^0$, by (9) and (13), each of them has at least $\alpha_1|V'_1| \geq 9$ neighbors in V'_1 . Thus, we choose distinct $z_L^1, z_L^2, z_N^1, z_N^2, z_N^3 \in V'_1$ such that

$$x_L^1 \sim z_L^1, z_L^1, \quad x_L^2 \sim z_L^2, z_L^2, \quad x_N \sim z_N^1, z_N^2, z_N^3.$$

(Note that x_L^1 and x_L^2 may be from V_{12}^0 , and therefore they may not have too many neighbors in V'_1 , we then choose z_L^1 and z_L^2 from V'_1 as their neighbors, respectively.)

By (14), we see that any two vertices in $G[V'_1]$ have at least $(1/3 - 16\beta)n_1 \geq 14$ neighbors in common. Thus, we can choose distinct vertices $z^{11}, z^{22}, z^{12}, v_1^R \in V'_1 - \{x_L^1, x_L^2, x_N, z_L^1, z_L^2, z_N^1, z_N^2, z_N^3\}$ such that

$$z^{11} \sim z_L^1, z_L^1, \quad z^{22} \sim z_L^2, z_L^2, \quad z^{12} \sim z^{11}, z^{22}, \quad v_1^R \sim z^{12}, z_N^1.$$

Furthermore, by (14) again, we have $\delta(G[V'_1]) \geq (2/3 - 8\beta)n_1 \geq 17$. Choose $z_1^1, z_2^2, z_N^{11} \in V'_1$ not chosen above such that

$$z_1^1 \sim z^{11}, z_2^2 \sim z^{22}, z_N^{11} \sim z_N^1.$$

Let T_{11} be the graph with

$$V(T_{11}) = \{x_L^1, x_L^2, x_N, z_N^1, z_L^1, z_L^2, z^{11}, z^{12}, z^{22}, z^2, z_N^2, z_N^3, v_1^R, z_1^1, z_2^2, z_N^{11}\}$$

and with edges indicated above except the edges $x_L^1 z_L^1$ and $x_L^2 z_L^2$. We see that T_{11} is a tree with v_1^R as the only degree 2 vertex, and $|V(T_{11})| = 17$ and $|L(T_{11})| = 9$. Notice that in T_{11} , z_L^1, x_L^1 and z_L^2, x_L^2 are leaves, and x_N is a non-leaf. Figure 3 gives a depiction of T_{11} .

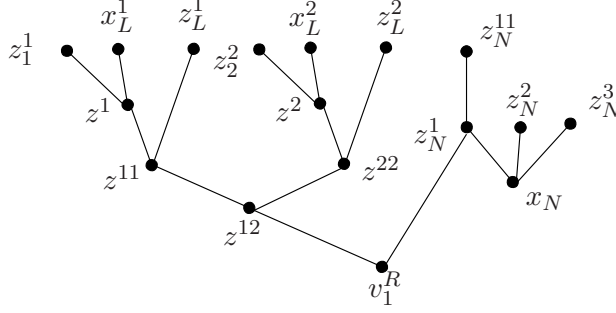


Figure 3: The tree T_{11}

Notice that the edges $x_L^1 z_L^1$ and $x_L^2 z_L^2$ are not used in T_{11} . We will first construct a HIST T_1 in $G[V_1^1 \cup V_{12}^0]$ containing T_{11} as a subgraph, then find a hamiltonian (z_L^1, z_L^2) -path on $L(T_1) - \{x_L^1, x_L^2\}$ by Lemma 3.6, finally by adding $x_L^1 z_L^1$ and $x_L^2 z_L^2$ to the path, we get a hamiltonian (x_L^1, x_L^2) -path on $L(T_1)$. The reason that we avoid using x_L^1 and x_L^2 is that when $x_L^1, x_L^2 \in V_{12}^0$, we may not be able to have the condition of Lemma 3.6 on $G[L(T_1)]$ in our final construction.

Then we initiate a HIT in $G[V_2']$ containing y_L^1, y_L^2 as leaves, and y_N as a non-leaf.

As $y_L^1, y_L^2, y_N \in V_2'$, by (15) and the fact that each two vertices from V_2' have at least $(1/3 - 2.2\alpha_1)n_2 \geq 7$ common neighbors implied from (15), we can choose distinct vertices

$$y^{12}, y_N^1, y_N^2, y_N^3, v_2^R \in V_2' - \{y_L^1, y_L^2, y_N\}$$

such that

$$y^{12} \sim y_L^1, y_L^2, \quad y_N \sim y_N^1, y_N^2, y_N^3, \quad v_2^R \sim y^{12}, y_N. \quad (15)$$

Let T_{21} be the graph with $V(T_{21}) = \{y_L^1, y_L^2, y_N, y^{12}, y_N^1, y_N^2, y_N^3, v_2^R\}$ and with $E(T_{21})$ described as in (15).

We see that T_{21} is a tree with v_2^R the only degree 2 vertex and $y_L^1, y_L^2 \in L(T_{21})$, $y_N \in S(T_{21})$ and

$$|V(T_{21}) \cap V_2'| = 8, \quad |L(T_{21}) \cap V_2'| = 5. \quad (16)$$

Denote

$$U_1 = V_1' - V(T_{11}), \quad U_2 = V_2' - V(T_{21}), \quad \text{and} \quad V_{12} = V_{12}^0 - V(T_{11}).$$

Step 4. Absorbing vertices in V_{12}^0

We may assume that $V_{12}^0 \neq \emptyset$. For otherwise, we skip this step. Let $|V_{12}| = n_{12}$ and $V_{12}^0 = \{x_1, x_2, \dots, x_{n_{12}}\}$.

Since $|V(T_{11})| = 17$, by (13), we get

$$\delta(V_{12}^0, U_1) > \alpha_1 |V_1'| - 17 \geq 3\alpha_2 |V_2| \geq 3|V_2 - V_2'| \geq 3|V_{12}^0|.$$

Thus, there is a claw-matching M_c from V_{12}^0 to U_1 centered in V_{12}^0 . For $i = 1, 2, \dots, n_{12}$, let x_{i1}, x_{i2} and x_{i3} be the three neighbors of x_i in M_c . If $n_{12} = 1$, let $T_a = M_c$, and we finish this step. Thus we assume $n_{12} \geq 2$.

By (14), each two vertices in V_1' have at least

$$(1/3 - 16\beta)n_1 \geq 6\alpha_2|V_{12}^0| + 17 \quad (17)$$

neighbors in common. The above inequality holds as $n_1 \geq 2n/5 - 2\beta n$, $|V_2| \leq 3n/5 + 2\beta n$ by (10), and we can assume that $18\alpha_2/5 + 106\beta/15 + 12\alpha_2\beta + 18/n - 32\beta^2 \leq 2/15$.

Thus, for each $i = 1, 2, \dots, n_{12}-1$, we can find distinct vertices $x_{13}^i, x_{23}^i, x_{i3}^3, x_{i+1,1}^3$ in $U_1 - V(M_c)$ such that

$$x_{13}^i \sim x_{i3}, x_{i+1,1}, \quad x_{23}^i \sim x_{13}^i, \quad x_{i3}^3 \sim x_{i3}, \quad x_{i+1,1}^3 \sim x_{i+1,1}. \quad (18)$$

Let T_a be the graph with $V(T_a) = V(M_c) \cup \{x_{13}^i, x_{23}^i, x_{i3}^3, x_{i+1,1}^3 : 1 \leq i \leq n_{12}-1\}$, and $E(T_a)$ including all edges indicated in (18) for all i and all edges in M_c . It is easy to see, by the construction, that T_a is a HIT with

$$|V(T_a) \cap U_1| = 7n_{12} - 4 \quad \text{and} \quad |L(T_a) \cap U_1| = 4n_{12} - 1.$$

Using (17) again, we can find $x_N^{11} \in U_1 - V(T_a)$ such that $x_N^{11} \sim v_1^R, x_{11}$, where $v_1^R \in V(T_{11})$ and $x_{11} \in V(T_a)$. By (14),

$$\delta[G[V_1']] \geq (2n+3)/5 - 2\beta n \geq 6\alpha_2|V_{12}^0| + 20,$$

since $|V_2| \leq 2n/5 + 2\beta n$, and we can assume that $2\beta - 12\alpha_2\beta - 18\alpha_2/5 - 21/n \leq 2/5$. So we can find distinct vertices $x_N^{12}, x_{11}^1 \in U_1 - V(T_a) - \{x_N^{11}\}$ such that $x_N^{12} \sim x_N^{11}, x_{11}^1 \sim x_{11}$.

Let T_1^1 be the graph with

$$V(T_1^1) = V(T_{11}) \cup V(T_a) \cup \{x_N^{11}, x_N^{12}, x_{11}^1\} \quad \text{and} \quad E(T_1^1) = E(T_{11}) \cup E(T_a) \cup \{x_N^{11}v_1^R, x_N^{11}x_{11}, x_N^{12}x_N^{11}, x_{11}^1x_{11}\}.$$

Then T_1^1 is a HIT such that

$$|V(T_1^1) \cap U_1| = 7n_{12} + 16 \quad \text{and} \quad |S(T_1^1) \cap U_1| = 3n_{12} + 7. \quad (19)$$

Denote $U_1' = U_1 - V(T_1^2)$ and $U_2' = U_2 - V(T_1^2)$.

Step 5. Completion of HITs T_1 and T_2

In this step, we construct a HIST T_i in $G[V_i']$ ($i = 1, 2$) containing T_1^i as an induced subgraph.

The following lemma guarantees the existence of a specified HIST in a graph with n vertices and minimum degree at least $(2/3 - \alpha')n$ for some $0 < \alpha' \ll 1$.

Lemma 4.6. *Let H be an n -vertex graph with $\delta(H) \geq (2/3 - \alpha')n$ for some constant $0 < \alpha' \ll 1$. Then H has a HIST T_H satisfying*

(i) T_H has a vertex v_R of degree at least $(2/3 - \alpha')n - 1$, and v_R can be chosen arbitrarily from $V(H)$;

(ii) $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$.

Proof. Let $v_R \in V(H)$ be an arbitrary vertex. If $n \pmod{2} \equiv \deg(v_R) + 1 \pmod{2}$, then we let $N_R = N_H(v_R)$. For otherwise, let N_R be a subset of $N(v_R)$ with $|N_H(v_R)| - 1$ elements. Let T_{v_R} be the star with $V(T_{v_R}) = \{v_R\} \cup N_R$ and $E(T_R) = E(\{v_R\}, N_R)$. Let $V_0 = V(H) - V(T_{v_R})$. By $\delta(H) \geq (2/3 - \alpha')n$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. By the choice of N_R , we have $|V_0| \equiv 0 \pmod{2}$. If $V_0 = \emptyset$, then let $T_H = T_{v_R}$. For otherwise, we claim as follows.

Claim 4.3. *Let $V_1 \subseteq V(H)$ be a subset with $|V_1| \geq (2/3 - \alpha')n - 1$ and $|V_1| \pmod{2} \equiv n \pmod{2}$. Then there exist two vertices from $V_0 = V(H) - V_1$ such that they have a common neighbor in V_1 .*

Proof of Claim 4.3. We assume that $|V_1| \leq (2/3 + 2\alpha')n$. For otherwise, $|V_0| < (1/3 - 2\alpha')n$. Since $\delta(H) \geq (2/3 - \alpha')n$, any two vertices of H have at least $(1/3 - 2\alpha')n$ neighbors in common. By $|V_0| < (1/3 - 2\alpha')n$, any two vertices from V_0 have a common neighbor from V_1 . We are done. Thus $|V_1| \leq (2/3 + 2\alpha')n$, and hence $|V_0| \geq (1/3 - 2\alpha')n \geq 3$. By the assumption that $|V_1| \geq (2/3 - \alpha')n - 1$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. This implies that $\deg(v_0, V_1) \geq (1/3 - 2\alpha')n - 2$ for each $v_0 \in V_0$. As $|V_0| \geq 3$ and $3((1/3 - 2\alpha')n - 2) > (2/3 + 2\alpha')n > |V_1|$ (provided that $8\alpha' + 6/n < 1/3$), we see that there must be two vertices from V_0 such that they have a neighbor in common in V_1 . \square

By Claim 4.3, there exist two vertices $v_0^{11}, v_0^{12} \in V_0$ such that they have a common neighbor in T_{v_R} . Adding v_0^{11} and v_0^{12} to T_{v_R} and two edges connecting them to one of their common neighbor in $V(T_{v_R})$. Let $T_{v_R}^1$ be the resulting graph. Then we see that $T_{v_R}^1$ is a HIT with $|V(T_{v_R}^1)| = |V(T_{v_R})| + 2$, and hence $(|V(T_{v_R}^1)| + 2) \pmod{2} \equiv n \pmod{2}$. Also $|V(T_{v_R}^1)| \geq |V(T_{v_R})| \geq (2/3 - \alpha')n - 1$. So we can use Claim 4.3 again to find another pair of vertices from $V_0 - \{v_0^{11}, v_0^{12}\}$ such that they have a common neighbor in $V(T_{v_R}^1)$. Adding the new pair of vertices and two edges connecting them to one of their common neighbor in $V(T_{v_R}^1)$ into $T_{v_R}^1$, we get a new HIT $T_{v_R}^2$. By repeating the above process another $l_0 = (|V_0| - 4)/2$ times, we get a HIT $T_{v_R}^{l_0}$. Let $T_H = T_{v_R}^{l_0}$. We claim that T_H has the required properties in Lemma 4.6. Notice first that $d_{T_H}(v_R) \geq (2/3 - \alpha')n - 1$. Then since T_H has v_R and at most $|V_0|/2$ distinct vertices as non-leaves and $|V_0| \leq (1/3 + \alpha')n + 1$, we see that $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$. \square

Let $H_1 = G[U_1' \cup \{v_R^1\}]$. Recall that v_R^1 is a non-leaf in T_1^1 . By (14) and (19), and by noticing that $n_{12} \leq |V_2 - V_2'| \leq \alpha_2|V_2| \leq 3\alpha_2 n_1/2$ (by (10)), we see that

$$\begin{aligned} \delta(H_1) &\geq (2/3 - 8\beta)n_1 - (7n_{12} + 19) \\ &\geq (2/3 - 8\beta)n_1 - 21\alpha_2 n_1/2 - 19 \\ &\geq (2/3 - 11\alpha_2)|V(H_1)|. \end{aligned} \tag{20}$$

Let $\alpha' = 11\alpha_2 \ll 1$ (by assuming $\alpha \ll 1$). By Lemma 4.6, we can find a HIT T_1' in H_1 with v_R^1 as the prescribed vertex in condition(i). It is easy to see that $T_1 := T_1^1 \cup T_1'$ is a HIST of $G[V_1' \cup V_{12}^0]$

and

$$\begin{aligned}
s_1 &= |S(T_1) \cap V'_1| = |S(T_1^1) \cap V'_1| + |S(T'_1) \cap V'_1| \\
&\leq 3n_{12} + 7 + (1/6 + 5.5\alpha_2)|V(H_1)| + 2 \text{ (by (19) and Lemma 4.6)} \\
&\leq 3n_{12} + 9 + (1/6 + 5.5\alpha_2)n_1 \\
&\leq (1/6 + 10.5\alpha_2)n_1 \text{ (by } n_{12} \leq 3\alpha_2 n_1/2\text{)}.
\end{aligned} \tag{21}$$

Let $H_2 = G[U'_2 \cup \{v_R^2\}]$. By (15) and (16), we see that

$$\delta(H_2) \geq (2/3 - 1.1\alpha_1)n_2 - 8 \geq (2/3 - 1.2\alpha_1)|V(H_2)|.$$

By letting $\alpha' = 1.2\alpha_1$, we can find a HIT T'_2 in H_2 with v_R^2 as the prescribed vertex in condition (i) of Lemma 4.6. Then $T_2 := T_1^2 \cup T'_2$ is a HIST of $G[V'_2]$. Also, notice that

$$\begin{aligned}
s_2 &= |S(T_2) \cap V'_2| = |S(T_1^2) \cap V'_2| + |S(T'_2) \cap V'_2| \\
&\leq 3 + (1/6 + 0.6\alpha_1)|V(H_2)| + 2 \\
&\leq (1/6 + 0.7\alpha_2)n_2,
\end{aligned} \tag{22}$$

where the last inequality holds by assuming $5/n_2 \leq 0.1\alpha_2$.

Step 6. Finding two long paths.

In this step, we first find a hamiltonian (z_L^1, z_L^2) -path P_1^1 in $G[L(T_1) - \{x_L^1, x_L^2\}]$; then find a hamiltonian (y_L^1, y_L^2) -path P_2 in $G[L(T_2)]$. Let $G_{11} = G[L(T_1) - \{x_L^1, x_L^2\}]$ and $n_{11} = |V(G_{11})|$. We will show that $\delta(G_{11}) > \frac{1}{2}n_{11}$. We may assume $s_1 \geq (1/6 - 8\beta)n_1 - 2$. For otherwise, if $s_1 < (1/6 - 8\beta)n_1 - 2$, then by (14), we get

$$\begin{aligned}
\delta(G_{11}) &\geq \delta(G[V'_1]) - s_1 - 2 \\
&\geq (2/3 - 8\beta)n_1 - ((1/6 - 8\beta)n_1 - 1 - 2) - 2 \\
&\geq \frac{1}{2}n_1 + 1 \geq \frac{1}{2}n_{11} + 1.
\end{aligned}$$

Hence, $s_1 \geq (1/6 - 8\beta)n_1 - 2$, implying that

$$n_{11} \leq (5/6 + 8\beta)n_1 + 2 \quad \text{and thus} \quad n_1 \geq \frac{n_{11} - 2}{5/6 + 8\beta}. \tag{23}$$

Hence, by (21)

$$\begin{aligned}
\delta(G_{11}) &\geq \delta(G[V'_1]) - s_1 - 2 \geq (2/3 - 8\beta)n_1 - (1/6 + 10.5\alpha_2)n_1 - 2 \\
&\geq (1/2 - 8\beta - 11\alpha_2)n_1 \geq \frac{1/2 - 2\beta - 11\alpha_2}{5/6 + 2\beta}(n_{11} - 2) > n_{11}/2,
\end{aligned}$$

the last inequality holds by assuming $3\beta + 11\alpha_2 + 2/n_{11} < 1/12$. By applying Lemma 4.6 on G_{11} , we find a hamiltonian (z_L^1, z_L^2) -path P_1^1 in G_{11} . Let $P_1 = P_1^1 \cup \{z_L^1 x_L^1, z_L^2 x_L^2\}$. We see that P_1 is a hamiltonian (x_L^1, x_L^2) -path on $L(T_1)$.

Let $G_{22} = G[L(T_2)]$ and $n_{22} = |V(G_{22})|$. We will show that $\delta(G_{22}) > n_{22}/2$. We may assume that $s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2$. For otherwise, if $s_2 < (1/6 - 1.1\alpha_1)n_2 - 2$, then by (15), we see that

$$\begin{aligned}\delta(G_{22}) &\geq \delta(G[V'_2]) - s_2 - 2 \\ &> (2/3 - 1.1\alpha_1)n_2 - ((1/6 - 1.1\alpha_1)n_2 - 2) - 2 \\ &> n_2/2 \geq n_{22}/2.\end{aligned}$$

Thus, $s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2$, implying that

$$n_{22} \leq n_1 - s_2 \leq (5/6 + 1.1\alpha_1)n_2 + 2 \quad \text{gives that} \quad n_2 \geq \frac{n_{22} - 2}{5/6 + 1.1\alpha_1}.$$

By (15) and (22),

$$\begin{aligned}\delta(G_{22}) &\geq \delta(G[V'_2]) - s_2 - 2 \\ &\geq (2/3 - 1.1\alpha_1)n_2 - (1/6 + 0.7\alpha_1)n_2 - 2 \\ &\geq (1/2 - 1.9\alpha_1)n_2 \geq \frac{1/2 - 1.9\alpha_2}{5/6 + 1.1\alpha_2}(n_{22} - 2) \\ &> n_{22}/2.\end{aligned}$$

The last inequality follows by assuming that $2.45\alpha_1 + 2/n_{11} < 1/12$. Hence, by Lemma 4.6, there is a hamiltonian (y_L^1, y_L^2) -path P_2 in G_{22} .

Step 7. Forming an SGHG

Let $T = T_1 \cup T_2 \cup \{x_N y_N\}$ and $C = P_1 \cup P_2 \cup \{x_L^1 y_L^1, x_L^2 y_L^2\}$. We see that T is a HIST of G with $L(T) = V(P_1) \cup V(P_2)$ and C is a cycle spanning on $L(T)$. Hence $H = T \cup C$ is an SGHG of G .

4.3 Proof of Theorem 4.3

Notice that the assumption of Extremal Case 2 implies that

$$|V_1| > (3/5 - \alpha)n \quad \text{and} \quad |V_2| \geq (2/5 - 2\beta)n.$$

We may assume that the graph G is minimal with respect to the number of edges. This implies that no two adjacent vertices both have degree larger than $(2n + 3)/5$. (For otherwise, we could delete any edges incident to two vertices both with degree larger than $(2n + 3)/5$.) We construct an SGHG in G step by step.

Step 1. Repartitioning

Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. Let

$$\begin{aligned}V'_2 &= \{v \in V_2 \mid \deg(v, V_1) \geq (1 - 3\alpha_1)|V_1|\}, \\ V'_0 &= \{v \in V_2 - V'_2 \mid \deg(v, V_1) \leq \alpha_1|V_2|/6\}, \\ V'_1 &= V_1 \cup V'_0, \quad V_{12}^0 = V_2 - V'_2 - V'_0.\end{aligned}$$

As $d(V_1, V_2) \geq 1 - 3\alpha$, the following holds,

$$\begin{aligned} (1 - 3\alpha)|V_1||V_2| &\leq e_G(V_1, V_2) = e_G(V_1, V'_2) + e_G(V_1, V_2 - V'_2) \\ &\leq |V_1||V'_2| + (1 - 3\alpha_1)|V_1||V_2 - V'_2|. \end{aligned}$$

The inequality implies that

$$|V_2 - V'_2| \leq \alpha_2|V_2|. \quad (24)$$

As a consequence of moving vertices in $V_2 - V'_2$ out from V_2 , by (24) we get

$$\begin{aligned} \delta(V_1, V'_2) &\geq (2n + 3)/5 - 2\beta n - \alpha_2|V_2| \\ &\geq (2n + 3)/5 - 6\beta|V_2| - \alpha_2|V_2| \\ &\geq (2n + 3)/5 - 2\alpha_2|V_2|, \end{aligned} \quad (25)$$

provided that $6\beta \leq \alpha_2$. And as a consequence of moving vertices in V'_0 to V_1 ,

$$\begin{aligned} \delta(V'_0, V'_2) &\geq \delta(G) - \Delta(V'_0, V_1) - \Delta(V'_0, V_2 - V'_2) \\ &\geq (2n + 3)/5 - \alpha_1|V_2|/6 - \alpha_2|V_2| \\ &\geq (2n + 3)/5 - \alpha_1|V_2|/3 \text{ (provided that } \alpha_2 \leq \alpha_1/6), \end{aligned} \quad (26)$$

and

$$\alpha_1|V_2|/6 < \delta(V_{12}^0, V'_1) < (1 - 3\alpha_1)|V_1|. \quad (27)$$

From (25) and (26), we have

$$\delta(V'_1, V'_2) \geq (2n + 3)/5 - \alpha_1|V_2|/3. \quad (28)$$

As

$$\delta(V'_2, V'_1) \geq (1 - 3\alpha_1)|V_1| \geq (1 - 3\alpha_1)(3/5 - \alpha)n > \lceil (2n + 3)/5 \rceil, \quad (29)$$

we get that

$$\deg(v'_1) = \lceil (2n + 3)/5 \rceil \quad (30)$$

for each $v'_1 \in V'_1$, by the minimality assumption of $e(G)$. Hence (28) and (30) give that

$$\Delta(G[V'_1]) \leq \alpha_1|V_2|/3. \quad (31)$$

Step 2. Finding a vertex v_2^* from V'_2 with large degree in V'_1

Let

$$e_{in} = e(G[V'_1]) \quad (32)$$

be the number of edges within V'_1 , notice that e_{in} maybe 0. Then

$$e_G(V'_1, V'_2 \cup V_{12}^0) = |V'_1|\lceil (2n + 3)/5 \rceil - 2e_{in}. \quad (33)$$

Let

$$d_{in} = e_{in}/|V'_1| \quad \text{and} \quad |n_0| = ||V'_2 \cup V_{12}^0| - \lceil (2n+3)/5 \rceil|. \quad (34)$$

By (31) and the definition of d_{in} in (34), we have

$$\lfloor d_{in} \rfloor \leq \alpha_1 |V_2|/6.$$

In fact, since $\Delta(V_1, V'_1) \leq \Delta(V_1, V_1) + \Delta(V_1, V'_0) \leq 2\beta n + |V'_0| \leq 2\beta n + \alpha_2 |V_2|$, and $\Delta(V'_0, V'_1) \leq \alpha_1 |V_2|/6 + \alpha_2 |V_2|$, more precisely, we have

$$\begin{aligned} 2d_{in} &= 2e_{in}/|V'_1| \leq (2\beta n + \alpha_2 |V_2|)|V_1|/|V'_1| + (\alpha_1 |V_2|/6 + \alpha_2 |V_2|)|V'_0|/|V'_1| \\ &\leq (2\beta n + \alpha_2 |V_2|) + \alpha_2 (\alpha_1 |V_2|/6 + \alpha_2 |V_2|) \quad (\text{as } |V'_0| \leq \alpha_2 |V_2| \text{ and } |V_1|, |V_2| \leq |V'_1|) \\ &\leq (6\beta + \alpha_2 + \alpha/6 + \alpha_2^2)|V_2| \quad (\text{as } \beta n \leq 3\beta |V_2|) \\ &\leq 2\alpha_2 |V_2| \quad (\text{provided that } 6\beta + \alpha/6 + \alpha_2^2 \leq \alpha_2). \end{aligned} \quad (35)$$

Set

Case A. $\lceil (2n+3)/5 \rceil - |V'_2 \cup V_{12}^0| = n_0 \geq 0$;

Case B. $|V'_2 \cup V_{12}^0| - \lceil (2n+3)/5 \rceil = n_0 \geq 1$.

We have

$$n_0 = \begin{cases} \lceil (2n+3)/5 \rceil - |V'_2 \cup V_{12}^0| \leq 2\beta n + \alpha_2 |V_2| \leq (6\beta + \alpha_2)|V_2| \leq 2\alpha_2 |V_2|, & \text{Case A,} \\ |V'_2 \cup V_{12}^0| - \lceil (2n+3)/5 \rceil \leq (2/5 + \alpha)n - \lceil (2n+3)/5 \rceil \leq \alpha n, & \text{Case B.} \end{cases} \quad (36)$$

Then in case A,

$$\begin{aligned} e_G(V'_1, V'_2 \cup V_{12}^0) &= |V'_1| \lceil (2n+3)/5 \rceil - 2e_{in} \quad (\text{by (30)}) \\ &= |V'_1| (|V'_2 \cup V_{12}^0| + n_0 - 2d_{in}) \\ &\geq |V'_2 \cup V_{12}^0| (|V'_1| + 1.4n_0 - 3.2d_{in}), \end{aligned}$$

as $1.4|V'_2 \cup V_{12}^0| \leq 1.4((2n+3)/5 + \alpha n) \leq (3/5 - \alpha)n < |V'_1|$ and $1.6|V'_2 \cup V_{12}^0| \geq 1.6((2n+3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n > |V'_1|$ provided that $2.4\alpha < 1/25$ and $5.2\beta + 2.6\alpha_2 \leq 1/25$ respectively. Since $e_G(V'_1, V'_2 \cup V_{12}^0) \leq |V'_2 \cup V_{12}^0||V'_1|$, we have $|V'_1| + 1.4n_0 - 3.2d_{in} \leq |V'_1|$, and thus $1.4n_0 \leq 3.2d_{in}$.

In Case B,

$$\begin{aligned} e_G(V'_1, V'_2 \cup V_{12}^0) &= |V'_1| \lceil (2n+3)/5 \rceil - 2e_{in} \quad (\text{by (30)}) \\ &= |V'_1| (|V'_2 \cup V_{12}^0| - n_0 - 2d_{in}) \\ &\geq |V'_2 \cup V_{12}^0| (|V'_1| - 1.6n_0 - 3.2d_{in}), \end{aligned}$$

as $1.6|V'_2 \cup V_{12}^0| \geq 1.6((2n+3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n > |V'_1|$ provided that $5.2\beta + 2.6\alpha_2 \leq 1/25$.

Let

$$d_l = \begin{cases} \lfloor 3.2d_{in} - 1.4n_0 \rfloor, & \text{if Case A,} \\ \lfloor 1.6n_0 + 3.2d_{in} \rfloor, & \text{if Case B.} \end{cases} \quad (37)$$

By (35) and (36), we see that

$$d_l \leq \begin{cases} 3.2\alpha_2|V_2|, & \text{if Case A,} \\ 6.4\alpha_2|V_2|, & \text{if Case B.} \end{cases} \quad (38)$$

Then there is a vertex v_2^* in $V_2' \cup V_{12}^0$ of degree at least $|V_1'| - d_l$. We will fix this vertex in what follows. In fact, such a vertex v_2^* is in V_2' by the facts that

$$\delta(V_{12}^0, V_1') < (1 - 3\alpha_1)|V_1| \quad \text{and} \quad |V_1'| - d_l \geq (1 - 3\alpha_1)|V_1|, \quad (39)$$

where $|V_1'| - d_l \geq (1 - 3\alpha_1)|V_1|$ holds because of (38).

Step 3. Finding a matching M within $G[\Gamma(v_2^*, V_1')]$

In this step, if $e_{in} \geq 1$, we first find a matching within $G[V_1']$ of size at least $e_{in}/(2 \triangle (G[V_1']))$. We assume this by giving the following lemma.

Lemma 4.7. *If G is a graph with maximum degree Δ , then G contains a matching of size at least $\frac{|E(G)|}{2\Delta}$.*

Proof. We use induction on $|V(G)|$. We may assume that the graph is connected. For otherwise, we are done by the induction hypothesis. Let $e = xy \in E(G)$ be an edge and $G' = G - \{x, y\}$. Since $|N_G(x) \cup N_G(y)| - |\{x, y\}| \leq 2(\Delta - 1)$, we have

$$e(G') \geq e(G) - 2(\Delta - 1) - 1 \geq e(G) - 2\Delta.$$

Hence, by the induction hypothesis, G' has a matching of size at least $\frac{e(G') - 2\Delta}{2\Delta} = \frac{e(G)}{2\Delta} - 1$. Adding e to the matching obtained in G' gives a matching of size at least $\frac{e(G)}{2\Delta}$ in G . \square

In case A, we take a matching in $G[V_1']$ of size at least $\max\{\lfloor 11d_{in} \rfloor, 11n_0\}$. This is possible because

$$\frac{e_{in}}{2 \triangle (G[V_1'])} \geq \frac{e_{in}}{2\alpha_1|V_1'|/3} = \frac{3d_{in}}{2\alpha_1} \geq 11d_{in}$$

provided that $\alpha \leq (\frac{3}{22})^3$, and

$$\begin{aligned} 2e_{in} &\geq |V_1'| \lceil (2n + 3)/5 \rceil - |V_1'| |V_2'| - (1 - 3\alpha_1) |V_1| |V_{12}^0| \\ &\geq |V_1'| \lceil (2n + 3)/5 \rceil - |V_1'| (\lceil (2n + 3)/5 \rceil - n_0 - |V_{12}^0|) - |V_1| |V_{12}^0| + 3\alpha_1 |V_1| |V_{12}^0| \\ &\geq |V_1'| n_0 + 3\alpha_1 |V_1| |V_{12}^0| \end{aligned} \quad (40)$$

implying that

$$\frac{e_{in}}{2 \triangle (G[V_1'])} \geq \frac{e_{in}}{2\alpha_1|V_1'|/3} \geq \frac{|V_1'| n_0 / 2}{2\alpha_1|V_1'|/3} \geq \frac{3n_0}{4\alpha_1} \geq 11n_0$$

provided that $\alpha \leq (\frac{3}{44})^3$.

By (37), $|V'_1| - \Gamma(v_2^*, V'_1) \leq d_l \leq \lfloor 3.2d_{in} \rfloor$, we can then choose a matching M from $\Gamma(v_2^*, V'_1)$ such that

$$|M| = \max\{\lfloor 7d_{in} \rfloor, 7n_0\}. \quad (41)$$

In case B, we take a matching in $G[V'_1]$ of size at least $\lfloor 8d_{in} \rfloor$. This is possible as

$$\frac{e_{in}}{\Delta(G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} = \frac{3d_{in}}{2\alpha_1} \geq \lfloor 8d_{in} \rfloor$$

provided that $\alpha \leq (\frac{3}{16})^3$.

By the second equality of (37), $|V'_1| - \Gamma(v_2^*, V'_1) \leq \lfloor 3.2d_{in} + 1.6n_0 \rfloor$. If $n_0 < 2d_{in}$, then $\lfloor 3.2d_{in} + 1.6n_0 \rfloor \leq \lfloor 7d_{in} \rfloor$. Thus, there is a matching M within $\Gamma(v_2^*, V'_1)$ such that

$$|M| = \begin{cases} \lfloor d_{in} \rfloor & \text{if } n_0 < 2d_{in}, \\ 0, & \text{if } n_0 \geq 2d_{in}. \end{cases} \quad (42)$$

We fix M for denoting the matching we defined in this step hereafter.

Step 4. Insertion

In this step, we insert vertices in V_{12}^0 into $V'_1 - V(M)$. Let $I = V_{12}^0 = \{x_1, x_2, \dots, x_I\}$, $U_1 = \Gamma(v_2^*, V'_1) - V(M)$, and $U_2 = V'_2$. Then (i)

$$\begin{aligned} \delta(I, U_1) &\geq \delta(I, V'_1) - |V(M)| - |V'_1 - \Gamma(v_2^*, V'_1)| \\ &\geq \alpha_1|V_2|/6 - \max\{\lfloor 7d_{in} \rfloor, 7n_0\} - \lfloor 1.6n_0 + 3.2d_{in} \rfloor, \\ &\geq \alpha_1|V_2|/6 - 20.4\alpha_2|V_2| \quad (\text{by (35) and (36)}) \\ &\geq 3\alpha_2|V_2| \geq 3|I| \quad (\text{provided that } 23.4\alpha_2 \leq \alpha_1/6), \end{aligned}$$

and from (28), we have (ii)

$$\delta(U_1, U_2 - \{v_2^*\}) \geq \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - 1 > \alpha_2|V_2| \geq |I|.$$

By condition (i), there is a claw-matching M_1 between I and U_1 centered in I . Suppose that $\Gamma(x_i, M_1) = \{x_{i0}, x_{i1}, x_{i2}\}$. We denote by P_{x_i} the path $x_{i1}x_ix_{i2}$. By (ii), there is a matching M_2 between $\{x_{i0} \mid 1 \leq i \leq |I|\}$ and $U_2 - \{v_2^*\}$ covering $\{x_{i0} \mid 1 \leq i \leq |I|\}$. So far, we get two matchings M_1 and M_2 .

Next we delete three types of edges not contained in

$$\left(\bigcup_{i=1}^{|I|} E(P_{x_i}) \right) \cup \{x_ix_{i0} : 1 \leq i \leq |I|\}.$$

Those edges include edges incident to a vertex in I , edges incident to a vertex in

$$\bigcup_{i=1}^{|I|} ((\Gamma(x_{i1}) - \Gamma(x_{i2})) \cup (\Gamma(x_{i2}) - \Gamma(x_{i1}))),$$

and one edge from the two edges connecting a vertex in $\Gamma(x_{i1}) \cap \Gamma(x_{i2})$ to both x_{i1} and x_{i2} , for each $i = 1, 2, \dots, |I|$.

For the resulting graph after the deletion of edges above, we contract each path P_{x_i} ($1 \leq i \leq |I|$) into a single vertex v_{x_i} . We call each v_{x_i} a *wrapped vertex* and call P_{x_i} the *preimage* of v_{x_i} . Denote by G^* the graph obtained by deleting and contracting the same edges as above, and let $U_2^* = V_2'$ and $U_1^* = V(G^*) - U_2^*$. (We will need the following degree condition in the end of this proof.) Since $|U_2^*| = |V_2'| \leq (2/5 + \alpha)n$, combining with (28), we have

$$\deg(v_{x_i}, U_2^*) \geq |\Gamma(x_{i1}, U_2^*) \cap \Gamma(x_{i2}, U_2^*)| - 1 \geq 2n/5 - \alpha_1|V_2|.$$

By the above inequality and (28), we get the first inequality below in (43). Since one edge from the two edges connecting a vertex in $\Gamma(x_{i1}) \cap \Gamma(x_{i2})$ to both x_{i1} and x_{i2} is deleted in G^* for each $i = 1, 2, \dots, |I|$, combining with (29), we have the second inequality as follows.

$$\begin{aligned} \delta(U_1^*, U_2^*) &\geq 2n/5 - \alpha_1|V_2|, \\ \delta(U_2^*, U_1^*) &\geq \delta(V_2', V_1') - 1 \geq (1 - 3\alpha_1)|V_1| - 1. \end{aligned} \quad (43)$$

Let U_1' and U_2' be the corresponding sets of U_1 and U_2 , respectively, after the contraction. Let T_W be the graph with

$$V(T_W) = \{x_{i0}, v_{x_i} : 1 \leq i \leq |I|\} \cup (V(M_2) \cap U_2) \quad \text{and} \quad E(T_W) = \{x_{i0}v_{x_i} : 1 \leq i \leq |I|\} \cup E(M_2).$$

By the construction,

$$|V(T_W) \cap U_1'| = |\{x_{i0}, v_{x_i} : 1 \leq i \leq |I|\}| = 2|I|, \quad |L(T_W) \cap U_1'| = |\{v_{x_i} : 1 \leq i \leq |I|\}| = |I|, \quad \text{and}$$

$$|V(T_W) \cap U_2'| = |L(T_W) \cap U_2'| = |V(M_2) \cap U_2'| = |I|.$$

Notice that T_W is a forest with $|I|$ components and each vertex x_{i0} ($1 \leq i \leq |I|$) has degree 2 in T_W . (We will make T_W connected in the end by connecting each x_{i0} to $v_{x_i}^*$.) See a depiction of this operation with $|I| = 1$ in Figure 4 below.

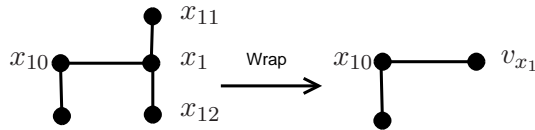


Figure 4: T_W with $|I| = 1$

Let $U_I^1 = (V'_1 - U_1) \cup U'_1 - V(T_W)$, $U_I^2 = U'_2 - V(T_W)$, and G_I the resulting graph with $V(G_I) = U_I^1 \cup U_I^2$. We have that

$$\begin{aligned} |U_I^1| &= |V'_1| - 3|I| = |V'_1| - 3n_{12}^0, & |U_I^2| &= |V'_2| - |I| = |V'_2| - n_{12}^0, \\ \delta(U_I^1, U_I^2) &\geq \delta(V'_1, V'_2) - n_{12}^0 \geq \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - n_{12}^0, \\ \delta(U_I^2, U_I^1) &\geq \delta(V'_2, V'_1) - 3n_{12}^0 \geq (1 - 3\alpha_1)|V_1| - 3n_{12}^0. \end{aligned} \quad (44)$$

Step 5. Matching Extension

In this step, in the graph G_I , we do some operation on the matching M found in Step 3. Notice that the vertices in M are unused in Step 4. Recall that $|M| \leq \max\{7n_0, \lceil 7d_{in} \rceil\}$. By $\lfloor d_{in} \rfloor \leq \alpha_2|V_2|$ from (35) and $n_0 \leq 2\alpha_2|V_2|$ from (36), we get

$$|M| \leq 14\alpha_2|V_2|. \quad (45)$$

Hence, $\delta(U_I^1, U_I^2 - \{v_2^*\}) \geq \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - n_{12}^0 - 1 \geq |M|$. Let V_M be the set of vertices containing exactly one end of each edge in M . Then there is a matching M' between V_M and $U_2 - \{v_2^*\}$ covering V_M . Let F_M be a forest with

$$V(F_M) = V(M) \cup (V(M') \cap U_2) \quad \text{and} \quad E(F_M) = E(M) \cup E(M').$$

Notice that

$$\begin{aligned} |V(F_M) \cap U_1| &= 2|M|, & |L(F_M) \cap U_1| &= |V(M) - V_M| = |M|, \\ |V(F_M) \cap U_2| &= |L(F_M) \cap U_2| = |M|. \end{aligned}$$

Notice that F_M has $|M|$ components, and all vertices in V_M has degree 2. (We will make F_M a HIT later on by connecting each vertex in V_M to the vertex $v_2^* \in U_2$) See Figure 5 for a depiction of F_M with $|M| = 3$.

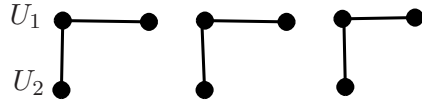


Figure 5: F_M with $|M| = 3$

Let

$$U_M^1 = U_I^1 - V(F_M) \quad \text{and} \quad U_M^2 = U_I^2 - V(F_M).$$

Notice that

$$\begin{aligned} |U_M^1| &= |U_I^1| - 2|M| = |V'_1| - 3n_{12}^0 - 2|M|, \\ |U_M^2| &= |U_I^2| - |M| = |V'_2| - n_{12}^0 - |M|, \end{aligned} \quad (46)$$

and

$$\begin{aligned}\delta(U_M^1, U_M^2) &= \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - n_{12}^0 - |M|, \\ \delta(U_M^2, U_M^1) &\geq (1-3\alpha_1)|V_1| - 3n_{12}^0 - 2|M|.\end{aligned}\tag{47}$$

Step 6. Distribute Remaining vertices in $U_M^1 - \Gamma(v_2^*, V_1')$

Let

We may assume $n_l \geq 1$. For otherwise, we skip this step. By (38), we have

$$n_l \leq \begin{cases} 3.2\alpha_2|V_2|, & \text{Case A,} \\ 6.4\alpha_2|V_2|, & \text{Case B.} \end{cases}\tag{48}$$

By $n_{12}^0 \leq \alpha_2|V_2|$ from (24) and $|M| \leq 14\alpha_2|V_2|$ from (45), we have (i)

$$\begin{aligned}\delta(U_M^1, U_M^2) &\geq \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - n_{12}^0 - |M| \geq \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \\ &\geq (1-3\alpha)|V_2| - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \quad (\text{as } \lceil (2n+3)/5 \rceil \geq (1-3\alpha)(2/5 + \alpha)n) \\ &\geq (1-3\alpha - \alpha_1/3 - 15\alpha_2)|V_2| \geq (1-\alpha_1)|V_2| \quad (\text{provided } 3\alpha + 15\alpha_2 \leq 2\alpha_1/3) \\ &\geq (1-\alpha_1)|U_M^2|.\end{aligned}\tag{49}$$

By (46) and (48), we have (ii)

$$\begin{aligned}|U_M^2| - 10\alpha_1|V_2| - \lceil n_l/10 \rceil - 1 &\geq |V_2'| - n_{12}^0 - |M| - 16\alpha_1|V_2| - 0.64\alpha_2|V_2| - 2 \\ &\geq (1-\alpha_2 - 14\alpha_2 - 10\alpha_1 - 0.64\alpha_2 - |V_2|/2)|V_2| \\ &\geq (1-11\alpha_1)|V_2| \quad (\text{provided } 15.64\alpha_2 + |V_2|/2 \leq \alpha_1) \\ &> 0 \quad (\text{provided } 11\alpha_1 < 1).\end{aligned}$$

Let $U_R = U_M^1 - \Gamma(v_2^*, V_1')$ and denote $\lceil \frac{|U_R|}{10} \rceil = l$. Suppose first that $|U_R| \geq 2$. We partition $U_R = U_{R_1} \cup U_{R_2} \cup \dots \cup U_{R_l}$ arbitrary such that each set contains at least 2 and at most $|U_R|/10$ vertices. Then by the conditions (i) and (ii), for each $1 \leq i \leq l$, there is a vertex $y_i \in U_2 - \{v_2^*\}$ which is common to all vertices in U_{R_i} , and is not used by any other U_{R_j} with $j \neq i$. Let T_R be the graph with

$$V(T_R) = U_R \cup \{y_i : 1 \leq i \leq l\} \quad \text{and} \quad E(T_R) = \{xy_i : x \in U_{R_i}, 1 \leq i \leq l\}.$$

Suppose now $|U_R| = 1$, let $U_R = \{x_R\}$. Choose $x'_R \in U_M^1 - U_R$ and $y_R \in U_M^2 - \{v_2^*\}$ be a vertex common to x_R and x'_R . Let T_R be a tree with

$$V(T_R) = \{x_R, x'_R, y_R\} \quad \text{and} \quad E(T_R) = \{x_R y_R, x'_R y_R\}.$$

By the construction,

$$|V(T_R) \cap U_M^1| = |L(T_R \cap U_M^1)| = \max\{|U_R|, 2\}, \quad |V(T_R) \cap U_M^2| = l, \quad \text{and} \quad |L(T_R \cap U_M^2)| = 0.$$

Notice that T_R is not connected when $|U_R| \geq 17$ and that T_R may have degree 2 vertices in $V(T_R) \cap U_M^2$. Later on, by joining each vertex in $T_R \cap U_M^2$ to a vertex of a tree, we will make the resulting graph connected, and thereby eliminating the possible degree 2 vertices in T_R . Let

$$U_R^1 = U_M^1 - V(T_R) \quad \text{and} \quad U_R^2 = U_M^2 - V(T_R).$$

Then we have

$$\begin{aligned} |U_R^1| &= |U_M^1| - n_l = |V_1'| - 3n_{12}^0 - 2|M| - \max\{2, n_l\}, \\ |U_R^2| &= |U_M^2| - \lceil n_l/10 \rceil = |V_2'| - n_{12}^0 - |M| - \lceil n_l/10 \rceil, \end{aligned} \quad (50)$$

and

$$\begin{aligned} \delta(U_R^1, U_R^2) &\geq \lceil (2n+3)/5 \rceil - \alpha_1 |V_2|/3 - n_{12}^0 - |M| - \lceil n_l/10 \rceil, \\ \delta(U_R^2, U_R^1) &= (1 - 3\alpha_1) |V_1| - 3n_{12}^0 - 2|M| - \max\{2, n_l\}. \end{aligned} \quad (51)$$

Let G_R be the subgraph of G induced on $U_R^1 \cup U_R^2$.

Step 7. Completion of a HIST in G_R

In this step, we find a HIST T_{main} in G_R such that

$$\begin{aligned} |L(T_{main}) \cap U_R^1| &= |L(T_W)|/2 + |L(F_M) \cap U_I^1| + |L(T_R) \cap U_M^1| = \\ |L(T_{main}) \cap U_R^2| &= |L(T_W)|/2 + |L(F_M) \cap U_I^2| + |L(T_R) \cap U_M^1|. \end{aligned}$$

By the construction of F_M and T_R , we have $|L(F_M) \cap U_I^1| = |L(F_M) \cap U_I^2|$ and $|L(T_R) \cap U_M^1| - |L(T_R) \cap U_M^2| = \max\{2, n_l\}$, respectively. So

$$|L(T_{main}) \cap U_R^2| - |L(T_{main}) \cap U_R^1| = \max\{2, n_l\}. \quad (52)$$

Notice that $v_2^* \in U_R^2$, v_2^* is adjacent to each vertex in U_R^1 , and $V_1' - \Gamma(v_2^*, V_1') \subseteq V(U_R^1)$. We now construct T_{main} step by step.

Step 7.1

Let T_{main}^1 be the graph with

$$V(T_{main}^1) = \{v_2^*\} \cup U_R^1 \quad \text{and} \quad E(T_{main}^1) = \{v_2^*x \mid x \in U_R^1\}.$$

To make the requirement of (52) possible, we need to make at least

$$\begin{aligned} d_3 &= |U_R^1| - |U_R^2| + \max\{2, n_l\}, \\ &= |V_1'| - |V_2'| - 2n_{12}^0 - |M| + \lceil n_l/10 \rceil \end{aligned} \quad (53)$$

vertices in U_R^1 with degree at least 3 in T_{main} , where the last inequality above follows from (50). Hereinafter, we assume that $\max\{2, n_l\} = n_l$. Since the proof for $\max\{2, n_l\} = 2$ follows the same idea, we skip the details.

Since all vertices in U_R^1 are included in T_{main}^1 and T_{main}^1 is connected, each vertex in T_{main}^1 needs to join to at least two distinct vertices from $U_R^2 - \{v_2^*\}$ to have degree no less than 3. Hence, to make a desired HIST T_{main} , it is necessary that

$$\begin{aligned}
d_{f*} &= |U_R^2| - 1 - 2d_3 \\
&= |V_2'| - n_{12}^0 - e_M - \lceil n_l/10 \rceil - 1 - 2d_3 \\
&= 3|V_2'| - 2|V_1'| + 3n_{12}^0 + |M| - 3\lceil n_l/10 \rceil - 1 \\
&\geq 0.
\end{aligned} \tag{54}$$

We show (54) is true, separately, for each of Case A and Case B. For Case A, notice that

$$|V_1'| = n - \lceil (2n+3)/5 \rceil + n_0 \quad \text{and} \quad |V_2'| = \lceil (2n+3)/5 \rceil - n_0 - n_{12}^0.$$

Hence,

$$3|V_2'| = 3\lceil (2n+3)/5 \rceil - 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V_1'| = 2n - 2\lceil (2n+3)/5 \rceil + 2n_0.$$

Thus,

$$\begin{aligned}
d_{f*} &= 5\lceil (2n+3)/5 \rceil - 2n - 5n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3\lceil n_l/10 \rceil - 1 \\
&\geq 2 - 5n_0 + |M| - 3\lceil \lfloor 3.2d_{in} \rfloor /10 \rceil \quad (\text{by } n_l \leq d_l \lfloor 3.2d_{in} - 1.4n_0 \rfloor \text{ from (37)}) \\
&= 2 - 5n_0 + \max\{7n_0, \lfloor 7d_{in} \rfloor\} - 3\lceil \lfloor 3.2d_{in} \rfloor /10 \rceil \\
&\geq 0.
\end{aligned}$$

Now we show (54) is true for case B. Notice that

$$|V_1'| = n - \lceil (2n+3)/5 \rceil + n_0 \quad \text{and} \quad |V_2'| = \lceil (2n+3)/5 \rceil + n_0 - n_{12}^0.$$

So

$$3|V_2'| = 3\lceil (2n+3)/5 \rceil + 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V_1'| = 2n - 2\lceil (2n+3)/5 \rceil + 2n_0.$$

Recall that $n_0 \geq 1$ in this case. We have

$$\begin{aligned}
d_{f*} &= 5\lceil (2n+3)/5 \rceil - 2n + n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3\lceil n_l/10 \rceil - 1 \\
&\geq 2 + n_0 + |M| - 3\lceil n_l/10 \rceil \\
&= 2 + n_0 + |M| - 3\lceil \lfloor 3.2d_{in} + 1.6n_0 \rfloor /10 \rceil \quad (\text{by } n_l \leq \lfloor 3.2d_{in} + 1.6n_0 \rfloor \text{ from (37)}) \\
&\geq \begin{cases} 2 + n_0 + \lfloor d_{in} \rfloor - \lfloor 9.2d_{in}/10 \rfloor - \lfloor 4.8n_0/10 \rfloor - 1 \geq 0, & \text{if } n_0 < 2d_{in}; \\ 2 + n_0 - \lfloor 9.2d_{in}/10 \rfloor - \lfloor 4.8n_0/10 \rfloor - 1 \geq 0, & \text{if } n_0 \geq 2d_{in}. \end{cases}
\end{aligned}$$

We now in Step 2 below show that there is a way to make exactly d_{f*} vertices in T_{main}^1 with degree 3 by joining each to two distinct vertices from $U_R^2 - \{v_2^*\}$.

Step 7.2

We first take $2d_3$ vertices from $U_R^2 - \{v_2^*\}$. For those $2d_3$ vertices, pair them up into d_3 pairs. We show that for each pair of vertices, they have at least d_3 common neighbors in U_R^1 . Using (51), $|M| \leq 14\alpha_2|V_2|$ from (45), $n_l \leq d_l \leq 6.4\alpha_2|V_2|$ from (38), we have

$$\begin{aligned}\delta(U_R^2, U_R^1) &\geq (1 - 3\alpha_1)|V_1| - 3n_{12}^0 - 2|M| - \max\{2, n_l\} \\ &\geq |V_1| - 3\alpha_1|V_1| - 3\alpha_2|V_2| - 28\alpha_2|V_2| - 6.4\alpha_2|V_2| \\ &\geq |V_1'| - |V_1 - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1|.\end{aligned}\tag{55}$$

Since $|U_R^1| \leq |V_1'|$, we know that any two vertices in U_R^2 have at least

$$\begin{aligned}n_c &= |V_1| - 2|V_1' - V_1| - 74.8\alpha_2|V_2| - 6\alpha_1|V_1| \\ &\geq (3/5 - \alpha)n - 76.8\alpha_2|V_2| - 6\alpha_1|V_1| \quad (\text{by } |V_1' - V_1| = |V_0'| \leq |V_2 - V_2'| \leq \alpha_2|V_2|) \\ &\geq 3n/5 - 10\alpha_1|V_1| \quad (\text{provided that } 76.8\alpha_2 + 3\alpha \leq 4\alpha_1)\end{aligned}$$

common neighbors in U_R^1 . On the other hand,

$$\begin{aligned}d_3 &= |V_1'| - |V_2'| - 2n_{12}^0 - |M| - \lceil n_l/10 \rceil \\ &\leq (3/5 - \alpha)n - (2n/5 - 2\beta n - |V_2 - V_2'|) + (1.6n_0 + 3.2\lfloor d_{in} \rfloor)/10 + 1 \\ &= n/5 - \alpha n + 2\beta n + |V_2 - V_2'| + (3.2\alpha_2|V_2| + 3.2\alpha_2|V_2|)/10 \quad (\text{by (35) and (36)}) \\ &\leq n/5 - \alpha n + 2\beta n + \alpha_2|V_2| + 0.64\alpha_2|V_2| \\ &\leq n/5 + 2\alpha_1|V_2| < n_c \quad (\text{provided } 12\alpha_1 < 2/5).\end{aligned}$$

Denote by $\{u_1^1, u_1^2\}, \{u_2^1, u_2^2\}, \dots, \{u_{d_3}^1, u_{d_3}^2\}$ the d_3 pairs of vertices from $U_R^2 - \{v_2^*\}$. Then by the above argument, we can choose d_3 distinct vertices say v_1, v_2, \dots, v_{d_3} from $L(T_{main}^1)$ such that $v_i \sim u_i^1, u_i^2$ for all $1 \leq i \leq d_3$.

Let T_{main}^2 be the graph with

$$V(T_{main}^2) = V(T_{main}^1) \cup \{u_i^1, u_i^2 : 1 \leq i \leq d_3\} \quad \text{and} \quad E(T_{main}^2) = E(T_{main}^1) \cup \{v_i u_i^1, v_i u_i^2 : 1 \leq i \leq d_3\}.$$

If $V(G_R) - V(T_{main}^2) = \emptyset$, we let $T_{main} = T_{main}^2$. For otherwise, we need one more step to finish constructing T_{main} .

Step 7.3

For the remaining vertices in $U_R^2 - V(T_{main}^2)$, we show that each of them has a neighbor in $S(T_{main}^2) \cap U_R^1$; that is, a neighbor in U_R^1 of degree 3 in $V(T_{main}^2)$. This is clear, as by (55), we have

$$\begin{aligned}\delta(U_R^2, U_R^1) &\geq |V_1'| - |V_1' - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1| \\ &\geq |U_R^1| - 38.4\alpha_2|V_2| - 3\alpha_1|V_1| \quad (\text{by } |V_1' - V_1| \leq |V_2 - V_2'| \leq \alpha_2|V_2|).\end{aligned}$$

Since $|S(T_{main}^2) \cap U_R^1| = d_3$, and

$$\begin{aligned}d_3 &= |V_1| - |V_2'| - 2n_{12}^0 - 2|M| + \lceil n_l/10 \rceil \\ &\geq (3/5 - \alpha)n - (2/5 + \alpha)n - 2\alpha_2|V_2| - 28\alpha_2|V_2| + 0.64\alpha_2|V_2| \\ &\geq n/5 - 2\alpha n - 29.36\alpha_2|V_2| \\ &> 38.4\alpha_2|V_2| + 3\alpha_1|V_1| \quad (\text{provided } 2\alpha + 67.76\alpha_2 + 3\alpha_1 < 1/5).\end{aligned}$$

Now, we join an edge between each vertex in $U_R^2 - V(T_{main}^2)$ and a neighbor of the vertex in $S(T_{main}^2) \cap U_R^1$. Let T_{main} be the resulting tree. By the construction procedure, it is easy to verify that T_{main} is a HIST of G_R .

Step 8. Connecting T_W , F_M , T_R , and $V(T_{main})$ into a connected graph

In this step, we connect T_W , F_M , T_R , and $V(T_{main})$ into a connected graph. Recall that each degree 2 vertex in T_W and F_M is a neighbor of v_2^* . We join an edge connecting v_2^* in $V(T_{main})$ and each degree 2 vertex in T_W and F_M . By the argument in step 7.3 above, we know each vertex in $V(T_R) \cap U_M^2$ has a neighbor in $S(T_{main}) \cap U_R^1$. Thus, we join an edge between each vertex in $V(T_R) \cap U_M^2$ to exactly one of its neighbor in $S(T_{main}) \cap U_R^1$. Let T^* be the final resulting graph. Notice that $I = V_{12}^0 = \{x_1, x_2, \dots, x_I\} \subseteq L(T^*)$ is the set of the wrapped vertices from Step 4. Recall that G^* is the graph obtained from G by deleting and contracting edges from Step 4. Then by the constructions of T_W , F_M , T_R , and T_{main} , we see that T^* is a HIST of G^* with $|L(T^*) \cap U_1^*| = |L(T^*) \cap U_2^*|$.

Step 9. Finding a cycle on $L(T^*)$

Denote

$$U_L^1 = L(T^*) \cap U_1^*, \quad U_L^2 = L(T^*) \cap U_2^* \quad \text{and} \quad G_L = G[E_G(U_L^1, U_L^2)].$$

Notice that G_L is a balanced bipartite graph. And

$$\begin{aligned} |S(T^*) \cap U_1^*| &= d_3 \leq n/5 + 2\alpha_1|V_2| \quad (\text{by (56)}) \\ |S(T^*) \cap U_2^*| &= 1 + \lceil n_l/10 \rceil \leq 2 + 0.64\alpha_2|V_2| \quad (\text{by } n_l \leq d_l \leq 6.4\alpha_2|V_2| \text{ from (38)}). \end{aligned}$$

Thus by (43),

$$\begin{aligned} \delta_{G^*}(U_L^1, U_L^2) &\geq 2n/5 - \alpha_1|V_2| - (2 + 0.64\alpha_2|V_2|) > 3n/10 > |U_L^2|/2 + 1, \\ \delta_{G^*}(U_L^2, U_L^1) &\geq (1 - 3\alpha_1)|V_1| - 1 - (n/5 + 2\alpha_1|V_2|) > n/3 > |U_L^1|/2 + 1. \end{aligned}$$

By Lemma 3.7, G_L contains a hamiltonian cycle C' .

Step 10. Unwrap vertices in $V(C') \cap \{v_{x_1}, v_{x_2}, \dots, v_{x_{|I|}}\}$

On C' , replace each vertex v_{x_i} with its preimage $P_{x_i} = x_{i1}x_ix_{i2}$ for each $i = 1, 2, \dots, |I|$. Denote the resulting cycle by C . Recall that $x_{i1}, x_{i2} \in \Gamma(v_2^*)$ by the choice of x_{i1} and x_{i2} . Let T be the graph on $V(G)$ with

$$E(T) = E(T^*) \cup \{v_2^*x_{i1}, v_2^*x_{i2} : i = 1, 2, \dots, |I|\}.$$

Then T is a HIST of G . Let $H = T \cup C$. Then H is an SGHG of G .

The proof of Extremal Case 2 is finished. ■

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